Real Exchange Rate and Terms of Trade
Obstfeld and Rogoff, Chapter 4
1 Introduction

- Multiple goods

- Role of relative prices
2 Price of non-traded goods with mobile capital

2.1 Model

- Traded goods prices obey Law of One Price (LOOP)

- Constant returns to scale production
  
  - Traded goods are the numeraire
    \[ Y_T = A_T F(K_T, L_T) \]
  
  - Non-traded goods
    \[ Y_N = A_N F(K_N, L_N) \]
• Labor
  
  – Mobility is perfect within country assuring identical wage across industries
  
  – Labor is completely immobile between countries
  
  – Total labor supply is fixed

\[ L = L_T + L_N \]
• Capital

  – Mobile across countries

  – Costlessly transform tradeables into capital and capital into tradeables

  – Cannot transform non-tradeables into capital or capital into non-tradeables

  – Capital must be in place one period before it is used
Firm optimization problem

- Present value of profits in the tradeables sector with no depreciation

\[
\sum_{s=t}^{\infty} \left( \frac{1}{1 + r} \right)^{s-t} \left[ A_{T,s} F(K_{T,s}, L_{T,s}) - w_s L_{T,s} - \Delta K_{T,s+1} \right] \\
\Delta K_{T,s+1} = K_{T,s+1} - K_{T,s}
\]
- FO conditions
  
  * $K_{T,s+1}$
    
    \[
    \left[ A_{T,s+1}F_K \left( K_{T,s+1}, L_{T,s+1} \right) + 1 \right] \frac{1}{1 + r} - 1 = 0
    \]
    
    - divide by $L_{T,s+1}$ and rearrange
      
      \[
      A_{T,s+1}f_k \left( k_{T,s+1} \right) = r
      \]

  * $L_{T,s}$
    
    \[
    A_{T,s}F_L \left( K_{T,s}, L_{T,s} \right) = w_s
    \]
use property of CRS that

\[ A_T F_L (K_T, L_T) = A_{T,s} \left[ f (k_T) - f' (k_T) k_t \right] \]

\[ A_{T,s} \left[ f (k_{T,s}) - f' (k_{T,s}) k_{T,s} \right] = w_s \]
Present value of profits in the non-tradeables sector with no depreciation

\[
\sum_{s=t}^{\infty} \left( \frac{1}{1 + r} \right)^{s-t} \left[ p_s A_{N,s} F\left( K_{N,s}, L_{N,s} \right) - w_s L_{N,s} - \Delta K_{N,s+1} \right]
\]

* FO conditions

\[
K_{N,s+1}
\]

\[
p_s A_{N,s+1} g_k \left( k_{N,s+1} \right) = r
\]

\[
L_{N,s}
\]

\[
p_s A_{N,s} \left[ g \left( k_{N,s} \right) - g' \left( k_{N,s} \right) k_{N,s} \right] = w_s
\]
• Equilibrium

  – Four equations in four unknowns, $k_T, k_N, w, p$

  – World interest rate $r$ is given and exogenous
2.2 Factor Price Frontier

- $A_T f_k (k_T) = r$ solves for $k_T$

$$k_T (A_T, r) = f'^{-1} \left( \frac{r}{A_T} \right) \quad \text{where } \frac{\partial k_T}{\partial r} < 0$$

- Use $A_T F_L (K_T, L_T) = w$ to solve for $w$

$$w (r, A_T) = A_T f [k_T (A_T, r)] - r k_T (A_T, r) \quad \text{where } \frac{\partial w}{\partial r} = -k_T (A_T, r)$$

- implies a negative relationship between the marginal product of labor and marginal product of capital in an industry, given $k_T$

- defines factor price frontier
• Given \( w \) can solve for \( k_N \) and \( p \) using equations in non-traded goods industry

- Graph with \( p \) on vertical axis and \( k_N \) on horizontal

- \( MPK = r \) graph

\[
pA_N g' (k_N) = r
\]

* as \( k_N \) increases, its marginal product falls requiring an increase in \( p \)

* \( MPK = r \) line is upward-sloping
- \( MPL = w \) graphs

\[ pA_N \left[ g(k_N) - g'(k_N) k_N \right] = w \]

* as \( k_N \) increases, the marginal product of labor increases requiring that \( p \) fall for fixed \( w \)

* \( MPL = w \) line is downward-sloping
• PPP will hold if countries have identical production functions and mobile capital

• Relative factor quantities and prices are independent of demand conditions
  – holds only if both goods are produced
  – if no traded goods are produced, then the wage cannot be determined in traded goods sector and need another equation for demand
2.3 Anticipated Shocks

- Anticipated in $t - 1$ to occur in $t$

- $A_T$ increases
  
  - MPL in tradeables increases increasing $w$
  
  - MPK in tradeables increases increasing $k_T$

  - Increase in $w$ shifts $MPL = w$ line upwards, raising $p$ and $k_N$
\( A_N \) increases

- \( MPK = r \) and \( MPL = w \) both shift downward in exact proportion to the increase in \( A_N \)

- \( p \) falls so that marginal product in terms of tradeables is unchanged
• Analytical results

– Zero profit conditions

\[ A_T f(k_T) = rk_T + w; \quad pA_N g(k_N) = rk_N + w \]

* Take natural logs and totally differentiate with respect to \( A_T, k_T, r \) and \( w \)

\[ \ln A_T + \ln f(k_T) = \ln [rk_T + w] \]

\[ \frac{dA_T}{A_T} + \frac{f'(k_T) dk_T}{f(k_T)} = \frac{rdk_T + dw + k_T dr}{rk_T + w} \]

• Substitute from the FO condition on capital

\[ A_T f_k(k_T) = r \]
and from zero profit condition to get

\[
\frac{dA_T}{A_T} + \frac{rdk_T}{A_T f(k_T)} = \frac{rdk_T + dw + k_T dr}{A_T f(k_T)}
\]

- Write as proportionate rates of change

\[
\frac{dA_T}{A_T} + \frac{k_T r}{A_T f(k_T)} \left( \frac{dk_T}{k_T} \right) = \frac{r k_T}{A_T f(k_T)} \left( \frac{dr}{r} \right)
\]

- Simplifying

\[
\frac{dA_T}{A_T} = \frac{w}{A_T f(k_T)} \left( \frac{dw}{w} \right) + \frac{r k_T}{A_T f(k_T)} \left( \frac{dr}{r} \right)
\]
· define labor’s share in traded goods as

\[ u_{LT} = \frac{w_{LT}}{Y_T} \]

\[ \hat{A}_T = u_{LT}\hat{w} + (1 - u_{LT})\hat{r} \]

* Perform similar steps on zero profit condition for non-traded goods with \( \hat{r} = 0 \)

\[ u_{LN} = \frac{w_{LN}}{p Y_N} \]

\[ \hat{p} + \hat{A}_N = u_{LN}\hat{w} \]
* Substitute $\hat{w}$ out to yield

$$\frac{\hat{p} + \hat{A}_N}{u_{LN}} = \frac{\hat{A}_T}{u_{Lt}}$$

* Solve for the rate of change of the relative price of non-tradeables

$$\hat{p} = \frac{u_{LN}}{u_{LT}} \hat{A}_T - \hat{A}_N$$

* For $\frac{u_{LN}}{u_{LT}} \geq 1$, faster productivity growth in tradeables raises the relative price of non-tradeables

  - the effect is stronger the more labor intensive are non-tradeables relative to tradeables
Effect of world interest rate increase on relative price of non-tradeables

\[ \hat{p} = \frac{1}{u_{LT}} (u_{KN} - u_{KT}) \hat{r} = \frac{1}{u_{LT}} (u_{LT} - u_{LN}) \hat{r} \]

since \( u_{KT} = 1 - u_{LT} \)

- An increase in the relative price of capital raises the relative price of the good that uses capital intensively
  - If tradeables are more capital intensive, then relative price of non-tradeables falls
  - Converse of Stolper-Samuelson theorem: change relative product prices benefits the factor used intensively in the industry with the increased relative price
2.4 Empirical Implications

- Facts
  
  \[ \frac{u_{LN}}{u_{LT}} \geq 1 \]

  - productivity growth in non-tradeables, including services, is slower than productivity growth in tradeables

- Prediction: rising relative price of non-tradeables is confirmed in the data

- Real exchange rate

  - home and foreign price index

  \[ P = 1^\gamma p^{1-\gamma} = p^{1-\gamma} \quad P^* = 1^\gamma (p^*)^{1-\gamma} = (p^*)^{1-\gamma} \]
- home price level relative to foreign price level

\[
\frac{\text{Home price}}{\text{Foreign price}} = \frac{P}{P^*} = \left(\frac{p}{p^*}\right)^{1-\gamma}
\]

* depends on the relative price of non-traded goods

- rate of change of home relative to foreign prices

\[
\hat{P} - \hat{P}^* = (1 - \gamma)(\hat{p} - \hat{p}^*) = (1 - \gamma)\left[\frac{u_{LN}}{u_{LT}}(\hat{A}_T - \hat{A}_T^*) - (\hat{A}_N - \hat{A}_N^*)\right]
\]

* home experiences real appreciation (increase in home relative to foreign prices, equivalently, home goods purchase more foreign goods) if relative technical progress in tradeables is greater than in non-tradeables

* independent of preferences and demand side
• Harrod-Balassa-Samuelson proposition: price levels tend to rise with country per capita income

  – rich countries become rich primarily through technical progress which is greater in tradeables than in non-tradeables implying higher price levels

• Japan and productivity growth

  – Post WWII rapid productivity growth in manufacturing and slow productivity growth in services yielded very high price levels

  – reverse in US
3 Consumption and Production in the Long Run (role of demand)

3.1 Assumptions

- Homothetic preferences such that the desired ratio between tradeables and non-tradeables depends only on relative price, not on the level of spending

- Steady state with constant national wealth in terms of the tradeable good

\[ Q \equiv B + K_T + K_N \equiv B + K \]
• Constant labor force of $L$

• No productivity growth

• No government consumption

3.2 Composition of tradeables and non-tradeables

• Zero profit in traded and non-traded goods where overbars denote steady state

\[
\bar{Y}_T = r\bar{K}_T + w\bar{L}_T = [rk_T (r) + w (r)] \bar{L}_T
\]

\[
p (r) \bar{Y}_N = r\bar{K}_N + w\bar{L}_N = [rk_N (r) + w (r)] (L - \bar{L}_T)
\]
• GDP line is the PPF between tradeables and non-tradeables when profit-
maximizing international capital flows are allowed

  – Use non-tradeables equation to solve for \( \bar{L}_T \) and substitute into the
  tradeables equation

\[
\bar{Y}_T = - \left[ \frac{r k_T(r) + w(r)}{r k_N(r) + w(r)} \right] p(r) \bar{Y}_N + [r k_N(r) + w(r)] L
\]

  – When \( \bar{Y}_N \) increases, \( \bar{Y}_T \) falls by more than \( p(r) \) since \( k_T > k_N \)

  * \( p \) is the marginal rate of transformation in an autarkic equilibrium

  * additional capital must be borrowed from abroad if the economy
  is to expand its capital-intensive traded sector with no rise in the
  rental-to-wage ratio
● GNP line is economy’s steady-state budget constraint

  – locus of best feasible steady-state consumptions of tradeables and non-tradeables, $\tilde{C}_T$ and $\tilde{C}_N$

    $$\tilde{C}_T = -p(r) \tilde{C}_N + w(r) L + r \tilde{Q}$$

  – as $\tilde{C}_N$ increases, $\tilde{C}_T$ falls by $p(r)$

● Income expansion path for a given relative price $p(r)$
• Explanation of diagram

  – Consumers are simultaneously on their income expansion path and their budget line, the GNP line
    * intersection of two determines $\bar{C}_T$ and $\bar{C}_N$

  – Domestic consumption of non-tradeables equals domestic production of non-tradeables
    
    \[ \bar{C}_N = \bar{Y}_N \]

    * Production of tradeables is above $\bar{C}_N$ on GDP line with $\bar{Y}_T > \bar{C}_T$, yielding a positive trade balance
Current account balance must be zero implying

- agents must pay interest on debt they accumulated to invest in production of tradeables

- income expansion path implies that the country prefers more tradeables than would be provided in autarky at the world interest rate

- therefore, agents must borrow to invest in tradeables production (since tradeables are relatively more capital intensive)

Flatter income expansion path would indicate a preference for non-tradeables and agents would lend abroad since non-tradeables are not capital intensive
• Higher steady-state wealth \((\bar{Q})\)
  
  – vertical upward shift of GNP (income) line by \(r d \bar{Q}\)
  
  – consumption of both goods rises and production of tradeables falls
  
  – to produce more non-tradeables and less tradeables, country must send some capital abroad
  
  – lower trade surplus and smaller interest payments on debt
• Higher government spending on tradeables \((\bar{G}_T)\)
  
  – GNP line shifts downward by \(\bar{G}_T\)
  
  – consumption of both goods falls
  
  – production of non-tradeable falls implying an increase in production of tradeables
  
  – larger current account surplus to pay interest on debt
Higher government spending on non-tradeables \((\tilde{G}_N)\)

- goods market equilibrium in non-traded goods
  \[
  \tilde{Y}_N = \tilde{C}_N + \tilde{G}_N
  \]

- to maintain fixed ratio of \(\frac{\tilde{C}_T}{\tilde{C}_N}\) income expansion path must shift right by the increase in \(\tilde{G}_N\)

- production of non-tradeables expands and production of tradeables contracts

- send capital abroad and have smaller current account surplus and smaller debt payments
3.3 Productivity Trends and the Size of the Traded Goods Sector

- Can we explain decline in manufacturing employment by a relatively more rapid rise in productivity in traded relative to non-traded sector?
  
  - Assume tastes are unchanged
  
  - Rise in productivity in traded should imply that can produce more goods with fewer workers in traded sector
  
  - Workers switch to non-traded, yielding a fall in employment in traded sector
• Model

– Cobb-Douglas production with fixed total labor supply $[L]$

$$Y_N = A_N K_N^\alpha L_N^{1-\alpha} = A_N k_N^\alpha L_N$$

– Factor shares

$$u_{KN} \equiv \frac{rK_N}{pY_N} = \alpha \quad u_{LN} = \frac{wL_N}{pY_N} = 1 - \alpha$$

– Take logs of the production function and differentiate

$$\hat{A}_N + \alpha \hat{k}_N + \hat{L}_N = \hat{Y}_N$$
Set productivity growth in non-traded sector to zero and solve for rate of growth of labor

\[ \hat{L}_N = \hat{Y}_N - \alpha \hat{k}_N \]

* implying that higher output in \( N \) requires higher employment in \( N \) unless capital intensity rises

* since non-traded goods consumption must equal non-traded goods output

\[ \hat{L}_N = \hat{C}_N - \alpha \hat{k}_N \]
• Derive demand for non-tradeables \((\hat{C}_N)\)

  – CES utility

  \[
  \left[ \gamma^{\frac{1}{\theta}} C_T^{\frac{\theta-1}{\theta}} + (1 - \gamma)^{\frac{1}{\theta}} C_N^{\frac{\theta-1}{\theta}} \right]^{\frac{\theta}{\theta-1}} 0 < \gamma < 1 \quad \theta > 0
  \]

  – Total spending

  \[
  Z = C_T + pC_N
  \]

  – Maximize utility subject to the constraint on total spending

  * \(C_T\)

  \[
  \frac{\theta}{\theta - 1} \left[ \gamma^{\frac{1}{\theta}} C_T^{\frac{\theta-1}{\theta}} + (1 - \gamma)^{\frac{1}{\theta}} C_N^{\frac{\theta-1}{\theta}} \right]^{-\frac{1}{\theta} - 1} \frac{1}{\theta} \gamma^{\frac{1}{\theta}} C_T^{-\frac{1}{\theta}} = \lambda
  \]
\* \( C_N \)

\[
\frac{\theta}{\theta - 1} \left[ \gamma \frac{1}{\theta} C_T^{\theta-1} + (1 - \gamma) \frac{1}{\theta} C_N^{\theta-1} \right]^{-\frac{1}{\theta}} \theta - 1 \frac{1}{\theta} (1 - \gamma) \frac{1}{\theta} C_N^{\theta-1} = \lambda p
\]

- Take the ratio

\[
\left( \frac{\gamma}{1 - \gamma} \right)^{\frac{1}{\theta}} \left( \frac{C_T}{C_N} \right)^{-\frac{1}{\theta}} = \frac{1}{p}
\]

\[
\left( \frac{\gamma}{1 - \gamma} \right) \left( \frac{C_N}{C_T} \right) = p^{-\theta}
\]

- where \( \theta \) is the elasticity of substitution
Demand for tradeables and non-tradeables: guess and verify

\[ C_T = \frac{\gamma Z}{\gamma + (1 - \gamma) p^{1-\theta}} \]
\[ C_N = \frac{p^{-\theta} (1 - \gamma) Z}{\gamma + (1 - \gamma) p^{1-\theta}} \]

* show that this guess satisfies ratio of FO conditions

* and sum satisfies spending constraint

Take logs of non-tradeables and differentiate

\[ \ln C_N = -\theta \ln p + \ln (1 - \gamma) + \ln Z - \ln (\gamma + (1 - \gamma) p^{1-\theta}) \]

- totally differentiate

\[ \hat{C}_N = \hat{Z} - \theta \hat{p} \frac{(1 - \theta)(1 - \gamma) p^{-\theta}}{\gamma + (1 - \gamma) p^{1-\theta}} dp = \hat{Z} - \theta \hat{p} \frac{(1 - \theta)(1 - \gamma) p^{1-\theta}}{\gamma + (1 - \gamma) p^{1-\theta}} \hat{p} \]
simplify by setting $p = 1$

$$
\hat{C}_N = \hat{Z} - \theta \hat{p} - (1 - \theta) (1 - \gamma) \hat{p}
$$

$$
= \hat{Z} - \hat{p} [(1 - \gamma) + \theta \gamma]
$$

* tradeables consumption

  • is increasing in total expenditure ($Z$)

  • and decreasing in the relative price of non-tradeables ($p$)

• Compute $\hat{Z}$

  – Expenditure equals output

  $$
  Z = wL + r\bar{Q}
  $$
– Take logs

\[ \ln Z = \ln (wL + rQ) \]

– Differentiate with respect to \( Z \) and \( w \)

\[ \hat{Z} = \frac{wL}{wL + rQ} \hat{w} \equiv \psi_L \hat{w} \]

where \( \psi_L \) is labor’s share

– Use relation between wage and technology from before

\[ \hat{A}_T = u_{LT} \hat{w} \]

– Solve for \( \hat{w} \) and substitute

\[ \hat{Z} = \frac{\psi_L}{u_{LT}} \hat{A}_T \]
• Use $\hat{p}$ from above with $\hat{A}_N = 0$

$$\hat{p} = \frac{u_{LN}}{u_{LT}} \hat{A}_T = \frac{1 - \alpha}{u_{LT}} \hat{A}_T$$

– using the Cobb-Douglas production function for non-tradeables

• Write expression for $\hat{C}_N$ as a function of exogenous productivity growth in tradeables ($\hat{A}_T$)

$$\hat{C}_N = \hat{Z} - \hat{p} [(1 - \gamma) + \theta \gamma]$$

$$= \frac{\psi_L}{u_{LT}} \hat{A}_T - [(1 - \gamma) + \theta \gamma] \frac{1 - \alpha}{u_{LT}} \hat{A}_T$$

$$= \left\{ \psi_L - (1 - \alpha) [(1 - \gamma) + \theta \gamma] \right\} \frac{\hat{A}_T}{u_{LT}}$$
• Use first order condition on capital to derive \( \hat{k}_N \)

  – FO condition

  \[ r = pA_N \alpha k_N^{1-\alpha} \]

  – Take logs

  \[ \ln r = \ln p + \ln A_N + \ln \alpha + (1 - \alpha) \ln k_N \]

  – Differentiate with \( \hat{r} = \hat{A}_N = 0 \)

  \[ \hat{k}_N = \frac{\hat{p}}{1 - \alpha} \]

  – Substitute for \( \hat{p} \)

  \[ \hat{k}_N = \frac{\hat{A}_T}{u_{LT}} \]
• Substitute $\hat{C}_N$ and $\hat{k}_N$ into expression for $\hat{L}_N$

$$\hat{L}_N = \hat{C}_N - \alpha \hat{k}_N = \{\psi_L - \alpha - (1 - \alpha) [(1 - \gamma) + \theta \gamma]\} \frac{\hat{A}_T}{u_{LT}}$$

– as $\hat{A}_T$ occurs, wages increase and incomes increase, raising the demand for tradeables and non-tradeables

– relative price of non-tradeables ($p$) increases, reducing demand for non-tradeables

– capital intensity in production of non-tradeables increases since $w/r$ increases

– if $\theta = 1$, $\hat{L}_N$ falls, and employment in tradeables rises

– reverse result only if
* very small $\theta$ so do not substitute our of nontradeables much as their relative price rises

* very large labor share ($\psi_L$) due to high national debt and low national income

- productivity growth in manufacturing (tradeables) does not explain the decline in manufacturing (tradeables) employment
4 Short-run Analysis with Current Account Imbalance

4.1 Consumer’s problem

- Consumption-based price index
  
  - Utility

\[ U_t = \sum_{s=t}^{\infty} \beta^{s-t} u(C_s) \]

* \( C \) is a linear homogeneous function of \( C_T \) and \( C_N \)

\[ C = \Omega(C_T, C_N) \]
* $C$ is an index of consumption, sometimes called real consumption

- Definition: $P$ (consumption-based price index) is the minimum expenditure $Z = C_T + pC_N$ such that $C = \Omega (C_T, C_N) = 1$, given $p$.

* $P$ measures the least expenditure in terms of tradeables which buys a unit of $C$ on which utility depends

- CES index

$$C = \left[ \frac{1}{\gamma \theta} C_T^{\theta - 1} + (1 - \gamma) \frac{1}{\theta} C_N^{\theta - 1} \right]^{\frac{\theta}{\theta - 1}}$$

- Write consumption index in terms of $Z$ using expressions for tradeable and non-tradeable demand derived earlier

$$C_T = \frac{\gamma Z}{\gamma + (1 - \gamma) p^{1-\theta}} \quad C_N = \frac{p^{-\theta} (1 - \gamma) Z}{\gamma + (1 - \gamma) p^{1-\theta}}$$
Substitute

\[ C = \left\{ \gamma^{1/\theta} \left[ \frac{\gamma Z}{\gamma + (1 - \gamma) p^{1-\theta}} \right]^{\theta-1/\theta} + (1 - \gamma)^{1/\theta} \left[ \frac{p^{-\theta} (1 - \gamma) Z}{\gamma + (1 - \gamma) p^{1-\theta}} \right]^{\theta-1/\theta} \right\}^{\theta/\theta-1} \]

- Since \( P \) is the minimum \( Z \) such that \( C = 1 \), set \( C = 1 \) and solve for \( Z \)

\[ 1 = \gamma^{1/\theta} \left[ \frac{\gamma Z}{\gamma + (1 - \gamma) p^{1-\theta}} \right]^{\theta-1/\theta} + (1 - \gamma)^{1/\theta} \left[ \frac{p^{-\theta} (1 - \gamma) Z}{\gamma + (1 - \gamma) p^{1-\theta}} \right]^{\theta-1/\theta} \]

- Factor \( Z \) out

\[ 1 = \left\{ \gamma^{1/\theta} \left[ \frac{\gamma}{\gamma + (1 - \gamma) p^{1-\theta}} \right]^{\theta-1/\theta} + (1 - \gamma)^{1/\theta} \left[ \frac{p^{-\theta} (1 - \gamma)}{\gamma + (1 - \gamma) p^{1-\theta}} \right]^{\theta-1/\theta} \right\} Z^{\theta-1/\theta} \]
– Collect terms on $\gamma$ and $1 - \gamma$ and simplify

\[
\left\{ \gamma \left[ \gamma + (1 - \gamma) p^{1-\theta} \right]^\frac{1-\theta}{\theta} + (1 - \gamma) p^{1-\theta} \left[ \gamma + (1 - \gamma) p^{1-\theta} \right]^\frac{1-\theta}{\theta} \right\} Z^{\frac{\theta-1}{\theta}}
\]

\[
= \left[ \gamma + (1 - \gamma) p^{1-\theta} \right]^\frac{1-\theta}{\theta} \left[ \gamma + (1 - \gamma) p^{1-\theta} \right] Z^{\frac{\theta-1}{\theta}}
\]

\[
= \left[ \gamma + (1 - \gamma) p^{1-\theta} \right]^\frac{1}{\theta} Z^{\frac{\theta-1}{\theta}}
\]

– Solve for $Z$

\[
Z^{\frac{1-\theta}{\theta}} = \left[ \gamma + (1 - \gamma) p^{1-\theta} \right]^\frac{1}{\theta}
\]

\[
Z = \left[ \gamma + (1 - \gamma) p^{1-\theta} \right]^\frac{1}{1-\theta}
\]
– Since the consumption-based price index is the \( Z \) for which we have solved

\[
P = \left[ \gamma + (1 - \gamma) p^{1-\theta} \right]^{\frac{1}{1-\theta}}
\]

- Total consumption in terms of \( P \)

\[
C = \frac{Z}{P}
\]

- \( \frac{Z}{P} \) is the ratio of spending, measured in tradeables, to the minimum price in tradeables of a single unit of the consumption index
• Demand functions

  – Tradeables

  \[ C_T = \frac{\gamma Z}{\gamma + (1 - \gamma) p^{1-\theta}} = \frac{\gamma PC}{P^{1-\theta}} = \gamma \left( \frac{1}{P} \right)^{-\theta} C \]

  – Nontradeables

  \[ C_N = \frac{p^{-\theta}(1 - \gamma) Z}{\gamma + (1 - \gamma) p^{1-\theta}} = \frac{(1 - \gamma) p^{-\theta} PC}{P^{1-\theta}} = (1 - \gamma) \left( \frac{p}{P} \right)^{-\theta} C \]

  – Demand for consumption of each type is proportional to real consumption with a coefficient that is an isoelastic function of the ratio of the good’s price (in terms of the numeraire) to the price index
• Price index for Cobb-Douglas case $\theta \longrightarrow 1$

$$P = \exp \left\{ \frac{\ln \left( \left[ \gamma + (1 - \gamma) p^{1-\theta} \right] \right)}{1 - \theta} \right\}$$

$$= \exp \left\{ \frac{\ln \left( \left[ \gamma + \exp \left\{ \ln (1 - \gamma) + (1 - \theta) \ln p \right\} \right] \right)}{1 - \theta} \right\}$$

- L'Hopital's rule: take derivative of numerator and derivative of denominator inside the exp term

$$\frac{\frac{1}{\gamma + (1 - \gamma) p^{1-\theta}} \left[ - \ln p \exp \left\{ \ln (1 - \gamma) + (1 - \theta) \ln p \right\} \right]}{1 - \theta}$$

* and take limit as $\theta \longrightarrow 1$

$$= \ln p \left( 1 - \gamma \right)$$
* take the results to the exp

\[ P = \exp \left( \ln p \left( 1 - \gamma \right) \right) = p^{1-\gamma} \]
4.2 Reformulate Consumer’s Problem

- Present value budget constraint
  \[
  \sum_{s=t}^{\infty} \left( \frac{1}{1 + r} \right)^{s-t} (C_{T,s} + p_s C_{N,s}) = (1 + r) Q_t + \sum_{s=t}^{\infty} \left( \frac{1}{1 + r} \right)^{s-t} (w_s L_s - G_s)
  \]

  \[
  C_{T,s} + p_s C_{N,s} = Z = P_s C_s
  \]

- Flow budget constraint
  \[
  Q_{s+1} - Q_s = r Q_s + w_s L_s - G_s - P_s C_s
  \]

- Solve flow budget constraint for consumption and substitute into utility
  \[
  U_t = \sum_{s=t}^{\infty} \beta^{s-t} u \left( \frac{Q_{s+1} - (1 + r) Q_s + w_s L_s - G_s}{P_s} \right)
  \]
- Differentiate with respect to $Q_{s+1}$ for $s \geq t$ to obtain Euler equation in consumption

$$\frac{u'(C_s)}{P_s} = (1 + r) \beta \frac{u'(C_{s+1})}{P_{s+1}}$$

- Define consumption based real interest rate as the own rate of interest on the consumption index

$$1 + r^C_{s+1} = (1 + r) \frac{P_s}{P_{s+1}}$$

- loan one unit of $C$ on date $s$ and receive $1 + r^C_{s+1}$ of $C$ on date $s + 1$

- differs from $r$ when expect change in relative price of tradeables, which determines $P$
• Rewrite Euler equation

\[ u'(C_s) = \left(1 + r_{s+1}^C\right) \beta u'(C_{s+1}) \]

• Optimal consumption

  – Rewrite intertemporal budget constraint in terms of date \( t \) real consumption by dividing both sides by \( P_t \)
\[
\text{left-hand side}
\]

\[
\sum_{s=t}^{\infty} \left( \frac{1}{1 + r} \right)^{s-t} \left( \frac{P_s C_s}{P_t} \right) \\
= \sum_{s=t}^{\infty} \left[ \frac{P_{t+1}}{(1 + r) P_t} \right] \left[ \frac{P_{t+2}}{(1 + r) P_{t+1}} \right] \cdots \left[ \frac{P_s}{(1 + r) P_{s-1}} \right] C_s \\
= \sum_{s=t}^{\infty} C_s \frac{1}{\prod_{v=t+1}^{s} (1 + r_v^C)}
\]
* right hand side

\[(1 + r) Q_t + \sum_{s=t}^{\infty} \left( \frac{1}{1 + r} \right)^{s-t} (w_s L_s - G_s)\]

\[= (1 + r) Q_t + \sum_{s=t}^{\infty} \frac{(w_s L_s - G_s)}{s} \prod_{v=t+1}^{s} \left(1 + r_v^C\right)\]

* define

\[R_{t,s}^C = \prod_{v=t+1}^{s} \left(1 + r_v^C\right)\]

* intertemporal budget constraint becomes

\[\sum_{s=t}^{\infty} R_{t,s}^C C_s = (1 + r) Q_t + \sum_{s=t}^{\infty} R_{t,s}^C (w_s L_s - G_s)\]
let utility be CRRA

\[ u(C) = \frac{C^{1-1/\sigma}}{1 - 1/\sigma} \]

* yielding an Euler equation as

\[ C_s^{-1/\sigma} = \left(1 + r_{s+1}^C\right) \beta C_{s+1}^{-1/\sigma} \]

* solving for future consumption in terms of initial consumption

\[ C_{s+1} = \left[\left(1 + r_{s+1}^C\right) \beta\right]^{\sigma} C_s \]

\[ C_{t+1} = \left[\left(1 + r_{t+1}^C\right) \beta\right]^{\sigma} C_t \]

\[ C_{t+2} = \left[\left(1 + r_{t+2}^C\right) \beta\right]^{\sigma} C_{t+1} = \left[\left(1 + r_{t+2}^C\right) \beta\right]^{\sigma} \left[\left(1 + r_{t+1}^C\right) \beta\right]^{\sigma} C_t \]
left-hand side of budget constraint becomes

\[
\sum_{s=t}^{\infty} R_{t,s}^{C} C_s = C_t \left[ 1 + R_{t,t+1}^{C} \left( 1 + r_{t+1}^{C} \right) \beta \right]^\sigma + R_{t,t+2}^{C} \left( 1 + r_{t+2}^{C} \beta^2 \right)^\sigma + \ldots \\
= C_t \left[ 1 + \left( R_{t,t+1}^{C} \right)^{1-\sigma} \beta^\sigma + \left( R_{t,t+2}^{C} \right)^{1-\sigma} \beta^{2\sigma} + \ldots \right] \\
= C_t \left[ \sum_{s=t}^{\infty} \left( R_{t,s}^{C} \right)^{1-\sigma} \beta^{\sigma(s-t)} \right]
\]
– substitute into the intertemporal budget constraint to yield a solution for consumption
\[
C_t = \frac{(1 + r) Q_t + \sum_{s=t}^{\infty} R_{t,s}^C (w_s L_s - G_s)}{\sum_{s=t}^{\infty} (R_{t,s}^C)^{1-\sigma} \beta^\sigma(s-t)}
\]

4.3 Equilibrium Current Account

- Since non-traded sector always clears, write model in terms of traded goods

- Intertemporal budget constraint in terms of traded goods assuming non-traded goods market clears
\[
\sum_{s=t}^{\infty} \left( \frac{1}{1 + r} \right)^{s-t} \left( C_{T,s} + I_s + G_{T,s} \right) = (1 + r) B_t + \sum_{s=t}^{\infty} \left( \frac{1}{1 + r} \right)^{s-t} \ Y_{T,s}
\]
• Traded goods Euler equation

  – begin with Euler equation for composite good $C$

    $C_{s+1} = \left[ \frac{(1 + r) P_s}{P_{s+1}} \right]^{\sigma} \beta^{\sigma} C_s$

  – relationship between $C_T$ and $C$

    $C_T = \gamma P^\theta C$

    $C = \frac{C_T}{\gamma P^\theta}$
- substitute into Euler equation

\[
\frac{C_{T,s+1}}{\gamma P_s^{\theta}} = \left[ \frac{(1 + r) P_s}{P_{s+1}} \right]^\sigma \beta^\sigma \frac{C_{T,s}}{\gamma P_s^{\theta}}
\]

- simplify

\[
C_{T,s+1} = (1 + r)^\sigma \beta^\sigma \left( \frac{P_s}{P_{s+1}} \right)^{\sigma - \theta} C_{T,s}
\]

* effect of inflation depends on an intertemporal and an intratemporal effect

- intertemporal: as inflation rises, the real interest rate on \( C \) falls, decreasing \( C_{t+1} \) relative to \( C_t \) with elasticity \( \sigma \)

- intratemporal: as \( P_{s+1} \) rises, the relative price of non-tradeables rises, increasing \( C_{T,t+1} \) with elasticity \( \theta \)
- Solve for traded goods consumption using budget constraint and traded goods Euler equation

\[ C_{T,t} = \frac{(1 + r) B_t + \sum_{s=t}^{\infty} \left( \frac{1}{1+r} \right)^{s-t} (Y_{T,s} - I_s - G_{T,s})}{\sum_{s=t}^{\infty} \left[(1 + r)^{\sigma-1} \beta^\sigma\right]^{s-t} \left( \frac{P_t}{P_s} \right)^{\sigma-\theta}} \]
- Current account

\[ CA_t = B_{t+1} - B_t = rB_t + Y_{T,t} - C_{T,t} - I_t - G_{T,t} \]

- Presence of non-tradeables affects path of the current account through effect of \( P \)

- Consider a rising value for \( p \) and \( P \) over time
  
  * Case 1: \( \sigma > \theta \)
    
    - Traded goods consumption is initially higher than in model without traded goods
    
    - Traded goods consumption falls over time due to low real interest rate
    
    - Initial current account deficit followed by a current account surplus
* Case 2: $\sigma < \theta$

  - Traded goods consumption is initially lower

  - Rises over time as substitute away non-traded goods whose relative price is rising
5 Terms of Trade in a Dynamic Ricardian Model

5.1 Model Assumptions

- Continuum of goods in the world, indexed by \( z \in [0, 1] \)

- Two countries: home and foreign

- Labor is only factor of production \( L (L^*) \)
• Countries have different technologies
  
  – Home produces $z$ with $a(z)$
  
  – Foreign produces $z$ with $a^*(z)$

• Utility

$$U_t = \sum_{s=t}^{\infty} \beta^{s-t} \log C_s \quad C = \exp \left[ \int_{0}^{1} \log c(z) \, dz \right]$$

• Numeraire is $z = 1$, so that wages and commodity prices are expressed in terms of good 1
5.2 Consumption based price index

- Problem
  
  - Minimize expenditure on goods
    \[
    \min_{c(z)} \int_0^1 p(z) c(z) \, dz
    \]

  - Subject to
    
    \[
    C = \exp \left[ \int_0^1 \log c(z) \, dz \right] = 1
    \]

  - Take logs of both sides
    
    \[
    \int_0^1 \log c(z) \, dz = 0
    \]
• Form Lagrangian

\[ L = \int_0^1 p(z) c(z) \, dz - \lambda \int_0^1 \log c(z) \, dz \]

• FO condition with respect to \( c(z) \)

\[ p(z) - \lambda \left( \frac{1}{c(z)} \right) = 0 \]

\[ p(z) c(z) = \lambda \]

- Every good receives equal weight in expenditure
• Solve for $\lambda$ using FO condition and constraint

$$\int_0^1 [\log \lambda - \log p(z)] \, dz = 0$$

$$\log \lambda = \int_0^1 [\log p(z)] \, dz$$

– raising both sides to the exp yields

$$\lambda = \exp \int_0^1 [\log p(z)] \, dz$$

• Consumption based price index

$$P = \min_{c(z)} \int_0^1 p(z) \, c(z) \, dz = \int_0^1 \lambda \, dz = \lambda = \exp \int_0^1 [\log p(z)] \, dz$$
• Demands

  – When $C = 1$

    \[ c(z) = \frac{\lambda}{p(z)} = \frac{P}{p(z)} \]

  – Since intratemporal preferences are homothetic, for other values of $C$

    \[ c(z) = \frac{PC}{p(z)} \]

  – Expenditure on any interval of goods $[z_1, z_2] \subset [0, 1]$ is

    \[ \int_{z_1}^{z_2} p(z) c(z) \, dz = (z_2 - z_1) PC \]
5.3 Wages, prices, and production

- Relative home labor productivity schedule

\[ A(z) = \frac{a^*(z)}{a(z)} \quad \text{where } A'(z) < 0 \]

- as \( z \) increases, \( \frac{\text{foreign labor}}{\text{domestic labor}} \) falls

- home is relatively good at producing on interval \([0, \tilde{z}]\)

- foreign is relatively good at producing on interval \([\tilde{z}, 1]\)
• Wages: produce good more cheaply at home when

\[ wa(z) < w^* a^*(z) \]

\[ \frac{w}{w^*} < \frac{a^*(z)}{a(z)} = A(z) \]

– goods produced more cheaply abroad when

\[ \frac{w}{w^*} > A(z) \]

– cut-off value for \( z \)

\[ \frac{w}{w^*} = A(\tilde{z}) \]
• Derive equilibrium wages

  – World goods market equilibrium

  \[ P(C + C^*) = wL + w^*L^* \]

  – If home produces \((0, z)\)

    * own demand for home goods is

    \[ zPC \]

    * foreign demand for home goods is

    \[ zPC^* \]
– Market clearing in market for home goods

\[ zP(C + C^*) = wL = z(wL + w^*L^*) \]

* Solve for

\[ \frac{w}{w^*} = \frac{z}{1 - z} \left( \frac{L^*}{L} \right) = B \left( z, \frac{L^*}{L} \right) \]

- Relative home wage rises with an increase in \( z \) and with an increase in relative foreign labor
• Comparative Statics: increase in $\frac{L^*}{L}$

  – shifts $B$ inward as need reduction in $z$ for unchanged $\frac{w}{w^*}$

  * relative home wages $\left(\frac{w}{w^*}\right)$ rise and $z$ falls

  – for all goods produced in home before and after, purchasing power of wage over each good is unchanged at $\frac{w}{p(z)} = a(z)$ is unchanged

  – similarly for all goods produced in foreign before and after, $\frac{w^*}{p(z)} = a^*(z)$ is unchanged

  – home real wages in terms of the goods foreign, produced before and after the increase in relative foreign labor supply, rise

\[
\frac{w'}{p(z)'} = \frac{w'}{a^*(z)w^{*h}} > \frac{w}{a^*(z)w^*} = \frac{w}{p(z)}
\]
− foreign’s real wage must fall in terms of goods initially produced at home and still produced at home

− wage in terms of goods which move from home to foreign

* these goods can be produced more cheaply in foreign than in home

\[ p(z)' = w^*' a^* (z) < w' a (z) \]

* using the two ends of the above equation

\[ \frac{w}{p(z)} = \frac{1}{a(z)} < \frac{w'}{p(z)'} \]

* home’s real wage in terms of goods that become imports rises because foreign costs now are lower than domestic costs

* get opposite for foreign
Summary

* rise in relative foreign labor supply lowers its real wage and raises home’s real wage making home relatively better off

* happens even as foreign produces a wider range of goods (increased competitiveness)

Terms of Trade = \( \frac{\text{Price of Exports}}{\text{Price of Imports}} \)

* any good home still exports

\[
\% \Delta p = \log p(z)' - \log p(z) = \log w' - \log w > 0
\]

* for goods that were originally foreign exports and still are

\[
\% \Delta p = \log p(z)' - \log p(z) = \log w^* - \log w^* < 0
\]
* goods that home used to produce which foreign now produces and exports

\[ w^* a^* (z) - wa^* (z) < 0 \]

* home country sees its terms of trade improve
5.4 Current Account

- Permanent disturbances do not affect current account

- Transitory ones do
  - Foreign productivity increase
    * Temporary CA surplus as agents save to smooth consumption after productivity returns to normal
    * Consumption is permanently higher and the CA is balanced
5.5 Transport cost and non-traded goods

- Assumptions
  - A fraction $\kappa$ of any good shipped between countries evaporates in transit
  - Cost of good produced at home is $wa(z)$
  - Cost of imported good is $\frac{w^*a^*(z)}{1 - \kappa}$
• Good will be produced at home if

\[
wa(z) < \frac{w^* a^*(z)}{1 - \kappa}
\]

\[
\frac{w}{w^*} < \frac{a^*(z)}{a(z)(1 - \kappa)} = \frac{A(z)}{1 - \kappa}
\]

• Foreign country will produce good locally if

\[
w^* a^*(z) < \frac{wa(z)}{1 - \kappa}
\]

\[
\frac{w}{w^*} > \frac{(1 - \kappa) a^*(z)}{a(z)} = (1 - \kappa) A(z)
\]
• Plot $\frac{A(z)}{1-\kappa}$ and $(1-\kappa)A(z)$ with $z$ on horizontal axis

  – Relative wages will determine range of goods produced in each country

  – Home will produce goods 0 to $z^H$

  – Foreign will produce goods $z^F$ to 1

  – Goods $z^F$ to $z^H$ are non-traded
• Prices of non-traded goods are different in the two countries

\[ p(z) = a(z)w \quad p^*(z) = a^*(z)w^* \]

• Prices of traded goods are no longer equal due to transport costs

  – home price of import must be higher than foreign price of same good

\[ \frac{w^*a(z)}{1 - \kappa} = \frac{p^*}{1 - \kappa} \]
• Real exchange rate

– Price index in terms of numeraire as units of good 1 delivered in foreign

\[ P = \exp \left\{ \int_0^{z^H} \log [w(a(z))] \, dz + \int_{z^H}^1 \log \left[ \frac{w^*a^*(z)}{1 - \kappa} \right] \, dz \right\} \]

\[ P^* = \exp \left\{ \int_0^{z^F} \log \left[ \frac{wa(z)}{1 - \kappa} \right] \, dz + \int_{z^F}^1 \log \left[ w^*(a^*(z)) \right] \, dz \right\} \]

– Real exchange rate is the ratio \( \frac{P}{P^*} = \)

\[ \exp \left\{ \int_0^{z^F} \log (1 - \kappa) \, dz + \int_{z^F}^{z^H} \log \left[ \frac{w(a(z))}{w^*(a^*(z))} \right] \, dz - \int_{z^H}^1 \log (1 - \kappa) \, dz \right\} \]

\[ \exp \left\{ \log (1 - \kappa) (z^F - 1 + z^H) + \int_{z^F}^{z^H} \log \left[ \frac{w(a(z))}{w^*(a^*(z))} \right] \, dz \right\} \]
* Depends on the ratio of non-traded goods prices in the two countries

* Also depends on the range of production

* Increase in $z_F$ reduces $\frac{P}{P^*}$ since $\log (1 - \kappa) < 0$
  
  · range of goods foreign must import increases
  
  · spending on transport cost increase
  
  · increasing foreign price relative to domestic price
Joint determination of relative wages and patterns of specialization

- World demand equals world supply

\[ PC + P^*C^* = wL + w^*L^* \]

- Equilibrium supply of home output \((wL)\) equals demand for it

\[
\begin{align*}
    wL &= z^HPC + z^FP^*C^* \\
    &= z^HPC + z^F(wL + w^*L^* - PC)
\end{align*}
\]

- Home trade balance is production less spending

\[ TB = wL - PC \]
– Solve for $\frac{w}{w^*}$

$$wL = z^H PC + z^F (wL + w^*L^* - PC)$$

$$= z^H PC + z^F (wL + w^*L^* + TB - wL)$$

$$= z^H PC + z^F (w^*L^* + TB)$$

$$wL = z^H (wL - TB) + z^F (w^*L^* + TB)$$

$$wL(1 - z^H) = z^F w^*L^* + TB (z^F - z^H)$$

$$w = \frac{z^F w^*L^* + TB (z^F - z^H)}{L (1 - z^H)}$$
\[
\frac{w}{w^*} = \frac{z^F L^*}{L \left(1 - z^H \right)} + \frac{TB \left(z^F - z^H \right)}{w^* L \left(1 - z^H \right)} \\
= \frac{L^*}{L \left(1 - z^H \right)} \left[ z^F + \frac{TB \left(z^F - z^H \right)}{w^* L^*} \right]
\]

- Since good 1 is the numeraire

\[
\frac{w^*}{a(1)^*} = \frac{p(1)^*}{a(1)^*} = \frac{1}{a(1)^*}
\]

- Substitute for \( w^* \)

\[
\frac{w}{w^*} = \frac{L^*}{L \left(1 - z^H \right)} \left[ z^F + \frac{TB \left(z^F - z^H \right)}{L^*/a(1)^*} \right]
\]
- Need two cut-off values, $z^H$ and $z^F$, both equal to $\frac{w}{w^*}$, and they are related by

\[(1 - \kappa) A(z^F) = A(z^H) / (1 - \kappa)\]

* Assume a specific functional form

\[A(z) = \exp (1 - 2z) = \frac{a^* (z)}{a(z)} = \frac{\exp (1 - z)}{\exp z}\]

* Substitute into relationship between $z^H$ and $z^F$

\[(1 - \kappa) \exp (1 - 2z^F) = \exp (1 - 2z^H) / (1 - \kappa)\]

  - Take logs

\[2 \log (1 - \kappa) = 1 - 2z^H - 1 + 2z^F\]

\[z^H = z^F - \log (1 - \kappa)\]
Substitute into the expression for \( \frac{w}{w^*} \) to get an expression in \( z^F \) alone

\[
\frac{w}{w^*} = \frac{L^*}{L \left(1 - z^F + \log(1 - \kappa)\right)} \left[ z^F + \frac{TB \log(1 - \kappa)}{L^*/a(1^*)} \right]
\]

* Note that \( \frac{w}{w^*} \), named \( \tilde{B}(z^F) \), is increasing in \( z^F \)

* Intersection of \( \tilde{B}(z^F) \) with \( \frac{A(z)}{1 - \kappa} \) and \( (1 - \kappa) A(z) \) determines \( z^F \) and \( z^H \)

* The less is produced in foreign (rightward movement along horizontal axis) the more is produced in home and the higher is relative demand for home labor and home relative wage
Permanent shock increasing $L^*$, setting $TB = 0$

$$\tilde{B} \left( z^F \right) = \frac{w}{w^*} = \frac{z^F L^*}{L \left( 1 - zH \right)}$$

- $\tilde{B} \left( z^F \right)$ shifts upwards
- home’s relative wage increases
- the range of goods produced at home is smaller and the range produced in foreign is larger
- due to its lower wage, foreign exports some goods previously non-traded
- goods previously exported by home become non-traded
- relative price of foreign goods falls so that trade balance is unchanged
even as home exports fewer goods and foreign exports more

- home benefits at foreign's expense
Classical transfer problem: Consider an increase in $TB$ from initial value of zero

- $TB$ up reduces $\frac{w}{w^*}$ shifting $\tilde{B}(z^F)$ down [remember $\log (1 - \kappa) < 0$]

- the increased trade balance transfers spending from home to foreign and foreign spends some on non-traded goods

- traded sector contracts as labor is drawn into nontradeables

- excess demand for foreign labor raises foreign relative wages and reduces range of goods foreign produces

- country with a positive trade balance has a lower terms of trade

* trade balance up reduces relative spending
* demand for non-tradeables falls reducing relative wages

* the fall in relative wages reduces relative prices
Permanent effects from a temporary increase in foreign productivity (fall in $a^* (z)$)

- $A(z)$ falls so $\frac{w}{w^*}$ falls

- Relative prices in foreign country fall,
  \[ \frac{wa(z)}{w^*a^*(z)} \] rises
  improving home terms of trade and making home better off

- Agents smooth consumption so foreign has trade surplus while home has trade deficit

- Range of production abroad rises and at home falls

- In the long run, home has a permanent trade balance surplus to pay
for the initial deficit

* transfer effects

* country with trade balance surplus has a lower terms of trade