Current Accounts in Open Economies
Obstfeld and Rogoff, Chapter 2
1 Consumption with many periods

1.1 Finite horizon of $T$

- Optimization problem

  - maximize

  $$U_t = u(c_t) + \beta (c_{t+1}) + \beta^2 u(c_{t+2}) + ... + \beta^T u(c_{t+T}) \quad \beta < 1$$
subject to $T + 1$ flow budget constraints

$$CA_t = B_{t+1} - B_t = Y_t + rB_t - C_t - G_t - I_t$$

$$CA_{t+1} = B_{t+2} - B_{t+1} = Y_{t+1} + rB_{t+1} - C_{t+1} - G_{t+1} - I_{t+1}$$

... 

$$CA_{t+T} = B_{t+T+1} - B_{t+T} = Y_{t+T} + rB_{t+T} - C_{t+T} - G_{t+T} - I_{t+T}$$

Present value budget constraint

* Add all current accounts, but retain individual bond terms from right-hand side (must be substituted out)
* Alternatively, subtract net foreign income to get trade balance

\[
B_{t+1} - (1 + r) B_t = TB_t = Y_t - C_t - G_t - I_t
\]

\[
B_{t+2} - (1 + r) B_{t+1} = TB_{t+1} = Y_{t+1} - C_{t+1} - G_{t+1} - I_{t+1}
\]

\[
\vdots
\]

\[
B_{t+T+1} - (1 + r) B_{t+T} = TB_{t+T} = Y_{t+T} - C_{t+T} - G_{t+T} - I_{t+T}
\]

* Multiply second line by \( \left( \frac{1}{1+r} \right) \), third by \( \left( \frac{1}{1+r} \right)^2 \), ending with multiplying final by \( \left( \frac{1}{1+r} \right)^T \) to take present values

* Sum present values of trade surpluses, noting that all terms with
bonds except first and last drop out

\[
\left( \frac{1}{1 + r} \right)^T B_{t+T+1} - (1 + r) B_t = \sum_{s=t}^{t+T} (Y_s - C_s - G_s - I_s) \left( \frac{1}{1 + r} \right)^{s-t}
\]

* Impose

\[
B_{t+T+1} = 0
\]

since no one would hold assets going into the end of the world, and no other country would let this country hold debt (because they would be holding assets)

\[
\sum_{s=t}^{t+T} (C_s + G_s + I_s) \left( \frac{1}{1 + r} \right)^{s-t} = (1 + r) B_t + \sum_{s=t}^{t+T} Y_s \left( \frac{1}{1 + r} \right)^{s-t}
\]

PV expenditure = net foreign assets + PV income
– Write utility substituting flow budget constraint for consumption

\[ C_t = Y_t + rB_t - G_t - I_t - (B_{t+1} - B_t) \]

\[ U_t = \sum_{s=t}^{t+T} \beta^{s-t} u \{(1 + r) B_s - B_{s+1} + A_s F(K_s) - (K_{s+1} - K_s) - G_s\} \]

– First order conditions (Euler equations)

* Bonds

\[ \frac{u'(C_s)}{u'(C_{s+1})} = \beta (1 + r) \]

* Capital

\[ \frac{u'(C_s)}{u'(C_{s+1})} = \beta \left[ 1 + A_{s+1} F'(K_{s+1}) \right] \]
* Together the two Euler equations imply

\[(1 + r) = 1 + A_{s+1}F' (K_{s+1})\]
1.2 Infinite Horizon

- Intertemporal budget constraint changes

  \[ \lim_{T \to \infty} \left( \frac{1}{1 + r} \right)^T \left( B_{t+T+1} - (1 + r) B_t \right) \]

  \[ = \sum_{s=t}^{\infty} (Y_s - C_s - G_s - I_s) \left( \frac{1}{1 + r} \right)^{s-t} \]

- impose

  \[ \lim_{T \to \infty} \left( \frac{1}{1 + r} \right)^T B_{t+T+1} = 0 \]

  requiring that debt not grow faster than the interest rate
* combines No Ponzi Game Condition (NPG) requiring that present-value assets be non-negative in the limit

* with optimality condition whereby agents would not choose to forego consumption so that present-value assets could be positive in the limit

\[
\sum_{s=t}^{\infty} (C_s + G_s + I_s) \left( \frac{1}{1 + r} \right)^{s-t} = (1 + r) B_t + \sum_{s=t}^{\infty} Y_s \left( \frac{1}{1 + r} \right)^{s-t}
\]

PV expenditure = net foreign assets + PV income

- intertemporal budget constraint implies an upper bound on current debt since smallest present value of consumption is zero

\[
\sum_{s=t}^{\infty} (Y_s - G_s - I_s) \left( \frac{1}{1 + r} \right)^{s-t} \geq - (1 + r) B_t
\]
requiring that debt \[ - (1 + r) B_t \] be less than the present value of net income (max country could repay if consumption were zero)

- Financial crisis
  - Either government or households want to borrow more than the upper bound on debt
  - Agents refuse to lend because they know the country cannot repay
  - Measure of debt burden
    * Since debt/GDP \( \left( \frac{B_s}{Y_s} \right) \) cannot grow forever, fix it, requiring
      \[
      B_{s+1} = B_s (1 + g),
      \]
    where \( g \) is rate of growth of output
From country flow budget constraint

\[ B_{s+1} - B_s = gB_s = Y_s + rB_s - G_s - I_s - C_s = rB_s + TB_s \]

\[ gB_s = rB_s + TB_s \]

\[ (g - r) B_s = TB_s \]

yielding trade balance surplus necessary to keep debt/GDP fixed

Assume have debt so \( B_s < 0 \). Either reduction in \( g \) or increase in \( r \) implies must run larger trade balance surplus to keep debt/GDP from growing
• Consumption

  – Assumptions

    * Infinite horizon, requiring infinite-horizon intertemporal budget constraint

    * $\beta (1 + r) = 1$, requiring constant consumption from Euler equation

  – Solve for constant consumption from intertemporal budget constraint

\[
\sum_{s=t}^{\infty} (C_s + G_s + I_s) \left( \frac{1}{1 + r} \right)^{s-t} = (1 + r) B_t + \sum_{s=t}^{\infty} Y_s \left( \frac{1}{1 + r} \right)^{s-t}
\]
\[ C \left( \frac{1 + r}{r} \right) = (1 + r) B_t + \sum_{s=t}^{\infty} (Y_s - G_s - I_s) \left( \frac{1}{1 + r} \right)^{s-t} \]

\[ C' = \frac{r}{1 + r} \left[ (1 + r) B_t + \sum_{s=t}^{\infty} (Y_s - G_s - I_s) \left( \frac{1}{1 + r} \right)^{s-t} \right] \]

- agent consumes the annuitized value of net wealth
- Benchmark consumption
  * \( G = I = 0 \)
  * \( Y = \bar{Y} \)

\[ C = r B_t + \bar{Y} \]
Consumption which varies over time \((\beta (1 + r) \neq 1)\)

- CRRA utility

\[
u(C) = \frac{C^{1-1/\sigma}}{1 - 1/\sigma}
\]

- Euler equation simplifies to

\[
C_{s+1} = C_s \beta^\sigma (1 + r)^\sigma
\]

* Taking this forward

\[
C_{s+2} = C_{s+1} \beta^\sigma (1 + r)^\sigma = C_s [\beta^\sigma (1 + r)^\sigma]^2
\]
– Consumption term for present-value budget constraint

\[
\sum_{s=t}^{\infty} C_s \left( \frac{1}{1+r} \right)^{s-t} = C_t + C_t \beta^\sigma (1 + r)^{\sigma-1} + C_t \left[ \beta^\sigma (1 + r)^{\sigma-1} \right]^2 + \ldots
\]

\[
= C_t \frac{1}{1 - \beta^\sigma (1 + r)^{\sigma-1}} \quad \text{for } \beta^\sigma (1 + r)^{\sigma-1} < 1
\]

– Define

\[ \nu = 1 - \beta^\sigma (1 + r)^\sigma \]

* Substituting

\[ 1 - \beta^\sigma (1 + r)^{\sigma-1} = \frac{\nu}{1 + r} + \frac{r}{1 + r} = \frac{\nu + r}{1 + r} \]

\[
\sum_{s=t}^{\infty} C_s \left( \frac{1}{1+r} \right)^{s-t} = C_t \frac{1 + r}{\nu + r}
\]
Solving for consumption using IBC yields

\[ C_t = \frac{\nu + r}{1 + r} \left[ (1 + r) B_t + \sum_{s=t}^{\infty} (Y_s - G_s - I_s) \left( \frac{1}{1 + r} \right)^{s-t} \right] \]

* \( \beta (1 + r) = 1, \nu = 0 \), consumption path is flat (benchmark case)

* \( \beta (1 + r) > 1, \nu < 0 \), agent is patient, and consumption begins smaller than benchmark and grows forever (eat less than annuitized value and bonds increase)

* \( \beta (1 + r) < 1, \nu > 0 \), agent is impatient, and consumption begins larger than benchmark and shrinks forever (eat more than annuitized value and bonds decrease)
2 Current Account with Infinite Horizon

2.1 Perfect Foresight

- Useful representation when $\beta (1 + r) = 1$

Define permanent value of a variable as

$$
\sum_{s=t}^{\infty} \left( \frac{1}{1 + r} \right)^{s-t} \tilde{X}_t = \sum_{s=t}^{\infty} \left( \frac{1}{1 + r} \right)^{s-t} X_s
$$

$$
\tilde{X}_t = \frac{r}{1 + r} \sum_{s=t}^{\infty} \left( \frac{1}{1 + r} \right)^{s-t} X_s
$$

the annuity value of the variable at the prevailing interest rate
Consumption

\[ C_t = rB_t + \tilde{Y}_t - \tilde{G}_t - \tilde{I}_t \]

Current account

\[ CA_t = rB_t + Y_t - C_t - I_t - G_t \]
\[ = Y_t - \tilde{Y}_t - (I_t - \tilde{I}_t) - (G_t - \tilde{G}_t) \]

* Current account imbalance occurs when variables deviate from permanent values

* Temporary increase in income implies a surplus as agents smooth consumption

* Temporary increase in spending implies a deficit as agents smooth consumption
Consumption tilting when $\beta (1 + r) \neq 1$

- Define

$$W_t = (1 + r) B_t + \sum_{s=t}^{\infty} (Y_s - G_s - I_s) \left( \frac{1}{1 + r} \right)^{s-t}$$

- Consumption becomes

$$C_t = \frac{\nu + r}{1 + r} W_t = r B_t + \tilde{Y}_t - \tilde{G}_t - \tilde{I}_t + \frac{\nu}{1 + r} W_t$$

- Current account becomes

$$CA_t = Y_t - \tilde{Y}_t - (I_t - \tilde{I}_t) - (G_t - \tilde{G}_t) - \frac{\nu}{1 + r} W_t$$

- Recall

$$\nu = 1 - \beta^\sigma (1 + r)^\sigma$$
For impatient (patient) agents $\nu > 0$, $(\nu < 0)$, and have current account deficit (surplus) even if variables take on permanent values
2.2 Stochastic Current Account

- Consumers maximize expected utility

\[ U_t = E_t \left\{ \sum_{s=t}^{\infty} \beta^{s-t} u(C_s) \right\} \]

subject to actual intertemporal budget constraint

\[ \sum_{s=t}^{\infty} (C_s + I_s) \left( \frac{1}{1+r} \right)^{s-t} = (1+r) B_t + \sum_{s=t}^{\infty} (Y_s - G_s) \left( \frac{1}{1+r} \right)^{s-t} \]

- Budget constraint must be obeyed with probability one
- Typically implies an endogenous upper bound on debt
- Euler equation

\[ u'(C_t) = (1 + r) \beta E_t u'(C_{t+1}) \]

- Quadratic utility

\[ u(C) = C - \frac{a_0}{2} C^2 \]

\[ u'(C) = 1 - a_0 C \]

\[ u''(C) = -a_0 \]

\[ u'''(C) = 0 \]

* Substitute into the Euler equation

\[ 1 - a_0 C_t = (1 + r) \beta E_t (1 - a_0 C_{t+1}) \]
* When $(1 + r) \beta = 1$, consumption is a random walk

$$C_t = E_t C_{t+1}$$

* Take expected value of budget constraint and solve for consumption,

$$C_t = \frac{r}{1 + r} \left[ (1 + r) B_t + E_t \sum_{s=t}^{\infty} (Y_s - G_s - I_s) \left( \frac{1}{1 + r} \right)^{s-t} \right]$$

replacing perfect foresight of future net income with expectation of future net income

* Certainty equivalence: agents make decisions under uncertainty by acting as if the future stochastic variables were equal to their means
* Problems with quadratic utility

1. Does not rule out $C$ becoming so large that $u'(C) < 0$

2. Does not rule out negative consumption

3. Ignores upper bound on debt
Consumption and the current account with quadratic utility, endowment economy, $G_s = I_s = 0$, and stationary output

- Stationary output has constant mean and variance
  \[ Y_t - \bar{Y} = \rho \left( Y_{t-1} - \bar{Y} \right) + \epsilon_t \quad E_{t-1} \epsilon_t = 0; \quad 0 \leq \rho < 1 \]

- Expected future output shows that effect of disturbance on output falls over time
  \[ \mathbb{E}_t \left( Y_{t+1} - \bar{Y} \right) = \rho \left( Y_t - \bar{Y} \right) = \rho^2 \left( Y_{t-1} - \bar{Y} \right) + \rho \epsilon_{t-1} \]

- Disturbance does not affect expected value of long-run output
  \[ \lim_{s \to \infty} \mathbb{E}_t \left( Y_{t+s} - \bar{Y} \right) = \lim_{s \to \infty} \rho^s \left( Y_t - \bar{Y} \right) = 0 \]
– Expected present value of output \((Y_t)\) becomes

\[
\bar{Y} \left( \frac{1 + r}{r} \right) + (Y_t - \bar{Y}) \left( 1 + \frac{\rho}{1 + r} + \left( \frac{\rho}{1 + r} \right)^2 + \ldots \right)
\]

\[
= \bar{Y} \left( \frac{1 + r}{r} \right) + (Y_t - \bar{Y}) \frac{1 + r}{1 + r - \rho}
\]

– Consumption

\[
E_t C_s = C_t = rB_t + \bar{Y} + (Y_t - \bar{Y}) \frac{r}{1 + r - \rho}
\]

\[
= rB_t + \bar{Y} + \left[ \rho (Y_{t-1} - \bar{Y}) + \epsilon_t \right] \frac{r}{1 + r - \rho}
\]

* Increase in \(\epsilon_t\) raises consumption permanently due to random walk

* Output reverts back to \(\bar{Y}\) at rate \(\rho\)

* Increase in output creates increase in savings so \(B_t\) rises to offset
fall in $Y_t$

\[ \text{MPC} = \frac{r}{1 + r - \rho} < 1 \]

- Current account

\[
CA_t = rB_t + Y_t - C_t \\
= Y_t - \bar{Y} - \left[ \rho(Y_{t-1} - \bar{Y}) + \epsilon_t \right] \frac{r}{1 + r - \rho} \\
= \left[ \rho(Y_{t-1} - \bar{Y}) + \epsilon_t \right] \left( 1 - \frac{r}{1 + r - \rho} \right) \\
= \left[ \rho(Y_{t-1} - \bar{Y}) + \epsilon_t \right] \left( \frac{1 - \rho}{1 + r - \rho} \right)
\]

* Current account is independent of $B_t$ since wealth affects $CA$ and $C$ identically
* Output shock has a transitory effect on the current account, which disappears once output has returned to its mean
Let output growth be stationary, giving output a unit root

\[ Y_t - Y_{t-1} = \rho (Y_{t-1} - Y_{t-2}) + \epsilon_t \quad 0 < \rho < 1 \quad E_{t-1}\epsilon_t = 0 \]

To simplify, set

\[ Y_{t-1} - Y_{t-2} = 0, \]

yielding

\[ Y_t = Y_{t-1} + \epsilon_t \]

Now, take time \( t \) expectations of future output

\[
E_t Y_{t+1} = Y_t + \rho (Y_t - Y_{t-1}) + E_t \epsilon_{t+1} \\
= Y_{t-1} + \epsilon_t + \rho \epsilon_t = Y_{t-1} + (1 + \rho) \epsilon_t
\]

\[
E_t Y_{t+2} = E_t Y_{t+1} + \rho E_t (Y_{t+1} - Y_t) \\
= Y_{t-1} + (1 + \rho) \epsilon_t + \rho^2 \epsilon_t = Y_{t-1} + (1 + \rho + \rho^2) \epsilon_t
\]
- Effect of current output shocks to future output grows over time

- If a shock raises output today, expect it to raise future output even more

- Consumption smoothing implies that consumption increases more than output creating CA deficit

- Note, additionally that

\[ E_t \Delta Y_{t+j} = \rho^j \epsilon_t \]
• Expression for the current account

  – Current account is the difference between income and permanent income

  \[ CA_t = Y_t - \tilde{Y}_t \]

  – Recall permanent income is

  \[ \tilde{Y}_t = \frac{r}{1 + r} E_t \sum_{j=0}^{\infty} \frac{Y_{t+j}}{(1 + r)^j}, \]

  \[ CA_t = Y_t - \frac{r}{1 + r} E_t \sum_{j=0}^{\infty} \frac{Y_{t+j}}{(1 + r)^j}. \]
- Recognize that

\[
Y_{t+1} = Y_t + \Delta Y_{t+1} \\
Y_{t+2} = Y_{t+1} + \Delta Y_{t+2} = Y_t + \Delta Y_{t+1} + \Delta Y_{t+2}
\]

Therefore, the sum we need can be written as

\[
\sum_{j=0}^{\infty} \frac{Y_t}{(1 + r)^j} + \sum_{j=1}^{\infty} \frac{\Delta Y_{t+1}}{(1 + r)^j} + \sum_{j=2}^{\infty} \frac{\Delta Y_{t+2}}{(1 + r)^j} + \ldots,
\]
yielding

\[
Y_t \left(\frac{1 + r}{r}\right) + \Delta Y_{t+1} \left(\frac{1}{r}\right) + \Delta Y_{t+2} \left(\frac{1}{r (1 + r)}\right) + \ldots
\]

We can write

\[
\tilde{Y}_t = Y_t + E_t \sum_{j=1}^{\infty} \frac{\Delta Y_{t+j}}{(1 + r)^j},
\]
Substituting, the current account becomes

\[ CA_t = -E_t \sum_{j=1}^{\infty} \frac{\Delta Y_{t+j}}{(1 + r)^j} = - \sum_{j=1}^{\infty} \left( \frac{\rho}{1 + r} \right)^j \epsilon_t = - \frac{1 + r}{1 + r - \rho} \epsilon_t \]
3 Current Account with Production and Uncertainty

3.1 Investment

- Optimization problem for household

$$\max_{B_{s+1}, K_{s+1}} U_t$$

$$= E_t \left\{ \sum_{s=t}^{\infty} \beta^{s-t} u \left[ (1 + r) B_s - B_{s+1} + A_s F(K_s) - (K_{s+1} - K_s) - G_s \right] \right\}$$
First order condition on bonds yields bond Euler equation with uncertainty

\[ \frac{\partial U_t}{\partial B_{t+1}} = E_t \left\{ u'(C_t)(-1) + \beta u'(C_{t+1})[1 + r] \right\} = 0 \]

\[ u'(C_t) = E_t \left\{ \beta u'(C_{t+1})[1 + r] \right\} \]

\[ 1 = E_t \left\{ \frac{\beta u'(C_{t+1})}{u'(C_t)}[1 + r] \right\} \]

First order condition on capital yields capital Euler equation with uncertainty

\[ \frac{\partial U_t}{\partial K_{t+1}} = E_t \left\{ u'(C_t)(-1) + \beta u'(C_{t+1}) \left[ A_{t+1}F'(K_{t+1}) + 1 \right] \right\} = 0 \]

\[ u'(C_t) = E_t \left\{ \beta u'(C_{t+1}) \left[ A_{t+1}F'(K_{t+1}) + 1 \right] \right\} \]
\[ 1 = E_t \left\{ \frac{\beta u'(C_{t+1})}{u'(C_t)} [A_{t+1} F'(K_{t+1}) + 1] \right\} \]

* Take expectation using \( E(xy) = E(x) E(y) + \text{cov}(x, y) \)

\[
1 = \frac{1}{1 + r} E_t [A_{t+1} F'(K_{t+1}) + 1] + \text{cov} \left( \frac{\beta u'(C_{t+1})}{u'(C_t)} ; [A_{t+1} F'(K_{t+1})] \right)
\]

\[
1 - \text{cov} \left( \frac{\beta u'(C_{t+1})}{u'(C_t)} ; [A_{t+1} F'(K_{t+1})] \right) = \frac{E_t [A_{t+1} F'(K_{t+1})]}{1 + r} + \frac{1}{1 + r}
\]

\[
\frac{r}{1 + r} - \text{cov} \left( \frac{\beta u'(C_{t+1})}{u'(C_t)} ; [A_{t+1} F'(K_{t+1})] \right) = \frac{E_t [A_{t+1} F'(K_{t+1})]}{1 + r}
\]

* Expected future marginal product of capital equals interest rate adjusted by a covariance term

* Differs from certainty equivalence (CEQ) by this covariance term
Sign of covariance term is likely negative because when $A_{t+1}$ is high, the marginal productivity of capital is high, $Y_{t+1}$ and $C_{t+1}$ are high, making $u'(C_{t+1})$ low.

Negative covariance adds a risk premium to capital because capital returns are high when consumption is already high.
Effect of productivity shock on current account with investment and stationary productivity

Assume

\[ A_{t+1} - \bar{A} = \rho \left( A_t - \bar{A} \right) + \epsilon_{t+1} = \rho \left( A_{t-1} - \bar{A} + \epsilon_t \right) + \epsilon_{t+1} \]

Taking expectations

\[ E_t \left( A_{t+1} - \bar{A} \right) = \rho \left( A_{t-1} - \bar{A} + \epsilon_t \right) \]

implying that a current increase in productivity increases expected future productivity

Current account response depends on response of consumption and investment

\[ CA_t = A_t F(K_t) - C_t - I_t \]
* Consumption increases less than output because output is temporarily high raising the current account

* Investment also increases since expected future productivity is high, reducing the current account

* Empirically current account is countercyclical, implying that the investment response must be large enough to reverse the response predicted by consumption alone
3.2 Precautionary savings

- Bond Euler equation with uncertainty

\[ 1 = (1 + r) \beta E_t \left\{ \frac{u'(C_{t+1})}{u'(C_t)} \right\} \]

- To have a precautionary motive, the third derivative of utility must be positive, implying that marginal utility is a convex function of consumption

\[ u' > 0, \quad u'' < 0, \quad u'''> 0 \]
- Illustrate with log utility

\[ u' = \frac{1}{C} > 0 \quad u'' = -\frac{1}{C^2} < 0, \quad u''' = \frac{1}{C^4} > 0 \]

- Euler equation with log utility

\[ 1 = (1 + r) \beta E_t \left\{ \frac{C_t}{C_{t+1}} \right\} = (1 + r) \beta C_t E_t \left\{ \frac{1}{C_{t+1}} \right\} \]

* In the absence of uncertainty \((1 + r) \beta = 1\), yields constant consumption over time

* However, with uncertainty, \((1 + r) \beta = 1\) does not imply that consumption is expected to be constant over time

\[ E_t \left\{ \frac{1}{C_{t+1}} \right\} > \frac{1}{E_t C_{t+1}} \]
* If \((1 + r) \beta = 1\),

\[
\frac{1}{C_t} = E_t \left\{ \frac{1}{C_{t+1}} \right\} > \frac{1}{E_t C_{t+1}},
\]

implying that \(E_t C_{t+1} > C_t\), such that consumption is expected to rise over time: precautionary saving

* In a closed economy with uncertainty, the a stationary equilibrium requires that \((1 + r) \beta < 1\) to assure that consumption is expected to be constant
- Wealth diminishes the precautionary motive as marginal utility flattens out as wealth and consumption increase

  * Precautionary savings yields a role for bonds (wealth) in determining the current account

  * As bonds increase with a current account surplus the precautionary motive weakens increasing spending and reducing the current account surplus

  * With precautionary savings, lose permanent effects of transitory shocks (through permanent effect on bonds)

  * Models linearized about the steady state lose the positive third derivative, lose the precautionary motive, and have the implication that a shock creates a permanent change in bonds and consumption
3.3 Current Account with Consumer Durables and No Uncertainty

- Utility

\[ U_t = \sum_{s=t}^{\infty} \beta^{s-t} [\gamma \log (C_s) + (1 - \gamma) \log D_s] \]

- Budget constraint with \( p_s \) the relative price of durables in terms of consumption

\[ B_{s+1} - B_s = rB_s + Y_s - C_s - p_s [D_s - (1 - \delta) D_{s-1}] - (K_{s+1} - K_s) - G_s \]
• Maximization problem using budget constraint to substitute for consumption

\[ U_t = \sum_{s=t}^{\infty} \beta^{s-t} (1 - \gamma) \log D_s + \sum_{s=t}^{\infty} \beta^{s-t} \gamma \log \left\{ (1 + r) B_s - B_{s+1} + Y_s - p_s \left[ D_s - (1 - \delta) D_{s-1} \right] \right\} \]

• First order condition with respect to bonds is bond Euler equation

\[ C_{s+1} = (1 + r) \beta C_s \]

  – First order condition with respect to durables

\[ \frac{\gamma p_s}{C_s} = \frac{1 - \gamma}{D_s} + \beta (1 - \delta) \frac{\gamma p_{s+1}}{C_{s+1}} \]

MU cost of acquiring durables = MU of immediate use + discounted

MU of selling what remains in one period
• Combine FO conditions to eliminate $C_{s+1}$

$$\frac{\gamma p_s}{C_s} = \frac{1 - \gamma}{D_s} + (1 - \delta) \frac{\gamma p_{s+1}}{(1 + r) C_s}$$

$$\text{MRS} \equiv \frac{(1 - \gamma) C_s}{\gamma D_s} = p_s - (1 - \delta) \frac{p_{s+1}}{1 + r} = \iota_s \equiv \text{user cost}$$

– where user cost is the price less the resale value of depreciated durables

• Intertemporal budget constraint with durables
– Durables term

\[ pt \left[ D_t - (1 - \delta) D_{t-1} \right] \]

\[ + p_{t+1} \left[ D_{t+1} - (1 - \delta) D_t \right] \left( \frac{1}{1 + r} \right) \]

\[ + p_{t+2} \left[ D_{t+2} - (1 - \delta) D_{t+1} \right] \left( \frac{1}{1 + r} \right)^2 \]

– Collecting terms on each \( D_t \)

\[ \sum_{s=t}^{\infty} \xi_s D_s \left( \frac{1}{1 + r} \right)^{s-t} - pt \left( 1 - \delta \right) D_{t-1} + \lim_{N \to \infty} p_{t+n} D_{t+N} \left( \frac{1}{1 + r} \right)^{t+N} \]
– Setting limit to zero, IBC becomes

\[
\sum_{s=t}^{\infty} (C_s + \iota_s D_s) \left( \frac{1}{1 + r} \right)^{s-t} = (1 + r) B_t + p_t (1 - \delta) D_{t-1} + \sum_{s=t}^{\infty} (Y_s - G_s - I_s) \left( \frac{1}{1 + r} \right)^{s-t}
\]

– Consumption and Durables when \( \beta (1 + r) = 1 \)

\[
C_t = \frac{\gamma r}{1 + r} \left[ (1 + r) B_t + p_t (1 - \delta) D_{t-1} + \sum_{s=t}^{\infty} (Y_s - G_s - I_s) \left( \frac{1}{1 + r} \right)^{s-t} \right]
\]

\[
D_t = \frac{(1 - \gamma) r}{\iota (1 + r)} \left[ (1 + r) B_t + p_t (1 - \delta) D_{t-1} + \sum_{s=t}^{\infty} (Y_s - G_s - I_s) \left( \frac{1}{1 + r} \right)^{s-t} \right]
\]

– If \( \iota_s \) is constant, then \( C_t \) and \( D_t \) are proportionate
– However, expenditures on the two, $C_t$ and $p_t [D_t - (1 - \delta) D_{t-1}]$ are not

– with $p$ and $\nu$ fixed, and $\delta = 0$, consumer buys all durables in beginning and never purchases them again

– with $\delta > 0$, replaces durables as they wear out

– when there are shocks to $p$ and $\nu$, get large changes in durables expenditure as move immediately to new equilibrium

– a change in demand for durables can lead to a large current account imbalance as durables expenditures change, yielding high current account volatility
4 Firms Distinct from Households with Certainty

4.1 Assumptions

- Production is homogenous of degree one, yielding CRS
  \[ Y = AF(K, L) \]

- \( L \) is fixed

- \( V_t \) is the price of a claim to a firm’s entire stream of future profits beginning on date \( t + 1 \)
• $x_{s+1}$ is share of domestic firm owned by the representative consumer at end of the period $s$

• $d_s$ is dividends per share on date $s$
4.2 Household Problem

- Consumer budget constraint

\[ B_{s+1} - B_s + V_s (x_{s+1} - x_s) = rB_s + d_s x_s + w_s L - C_s - G_s \]

- Optimization problem using budget constraint to substitute for consumption

\[ U_t = \sum_{s=t}^{\infty} \beta^{s-t} u [-B_{s+1} + (1 + r) B_s - V_s (x_{s+1} - x_s) + d_s x_s + w_s L - G_s] \]

- First order condition with respect to \( x_{s+1} \)

\[ \frac{\partial U_t}{\partial x_{s+1}} = \left\{ u' (C_s) (-V_s) + \beta u' (C_{s+1}) (d_{s+1} + V_{s+1}) \right\} = 0 \]
\[ 1 + r = \frac{u'(C_s)}{\beta u'(C_{s+1})} = \frac{d_{s+1} + V_{s+1}}{V_s} \]

- returns on equity are dividend plus value of asset relative to initial value of the asset

- in equilibrium with no uncertainty returns on all assets must be equal
Reformulate individual budget constraint

- Define $Q_{s+1}$ as total financial wealth going into period $s+1$

$$Q_{s+1} = B_{s+1} + V_s x_{s+1}$$

- FO condition, and equivalently arbitrage across asset returns, implies

$$(1 + r) V_s = d_{s+1} + V_{s+1}$$

- Solving for dividends

$$d_{s+1} = (1 + r) V_s - V_{s+1}$$
Copy agent budget constraint from above and rewrite

\[ B_{s+1} - B_s + V_s (x_{s+1} - x_s) = rB_s + d_s x_s + w_s L - C_s - G_s \]

\[ Q_{s+1} - Q_s - V_s x_s + V_{s-1} x_s = rQ_s - rV_{s-1} x_s + d_s x_s + w_s L - C_s - G_s \]

- collect terms on \( x_s \)

\[ d_s - [(1 + r) V_{s-1} - V_s] = 0 \]

- use the definition of dividends from above

\[ Q_{s+1} - Q_s = rQ_s + w_s L - C_s - G_s \quad \text{for } s > t \]

- holds only if arbitrage allows equality of returns to stocks and bonds

- does not hold in event of unanticipated shock which invalidates equality
– Sum present values of budget constraints

\[
Q_{t+1} - [(1 + r) B_t + d_t x_t + V_t x_t + w_t L - C_t - G_t] = 0
\]

\[
+ \frac{1}{1 + r} \left\{ Q_{t+2} - (1 + r) Q_{t+1} + w_{t+1} L - C_{t+1} - G_{t+1} \right\} = 0
\]

\[
+ \left( \frac{1}{1 + r} \right)^2 \left\{ Q_{t+3} - (1 + r) Q_{t+2} + w_{t+2} L - C_{t+2} - G_{t+2} \right\} = 0
\]

+ ....

\[
\left( \frac{1}{1 + r} \right)^{N-1} Q_{t+N}
\]

\[
- \left[ (1 + r) B_t + d_t x_t + V_t x_t + \sum_{s=t}^{N-1} (w_s L - C_s - G_s) \left( \frac{1}{1 + r} \right)^{t-s} \right]
\]

– take limit as \( N \rightarrow \infty \) and set the limit term to zero implying PV consumption equals PV labor income net of taxes plus initial assets
with interest and dividends

\[ \sum_{s=t}^{\infty} C_s \left( \frac{1}{1 + r} \right)^{t-s} = (1 + r) B_t + d_t x_t + V_t x_t + \sum_{s=t}^{\infty} (w_s L - G_s) \left( \frac{1}{1 + r} \right)^{t-s} \]
4.3 Value of the firm and stock prices

- Asset pricing equation is the FO condition with respect to $x_{t+1}$

$$V_t = \frac{d_{t+1} + V_{t+1}}{1 + r}$$

- Solve forward

$$V_{t+1} = \frac{d_{t+2} + V_{t+2}}{1 + r}$$

$$V_t = \frac{d_{t+1}}{1 + r} + \frac{1}{1 + r} \left[ \frac{d_{t+2} + V_{t+2}}{1 + r} \right]$$

$$V_t = \sum_{s=t+1}^{\infty} d_s \left( \frac{1}{1 + r} \right)^{s-t} + \lim_{T \to \infty} \left( \frac{1}{1 + r} \right)^T V_{t+T}$$
– Impose limit term equals zero requiring that stock value not grow faster than interest rate

– Value of stocks today is the present value of future dividends
4.4 Behavior of the Firm

- Dividends

\[ d_s = A_s F(K_s, L_s) - w_s L_s - (K_{s+1} - K_s) \]

- Value of the firm

\[ V_t = \sum_{s=t+1}^{\infty} \left( \frac{1}{1 + r} \right)^{s-t} \left[ A_s F(K_s, L_s) - w_s L_s - (K_{s+1} - K_s) \right] \]

- Firm maximizes present value of dividends

\[ V_t + d_t = \sum_{s=t}^{\infty} \left( \frac{1}{1 + r} \right)^{s-t} \left[ A_s F(K_s, L_s) - w_s L_s - (K_{s+1} - K_s) \right] \]
\begin{itemize}
  \item FO condition with respect to $K_{s+1}$

  \[
  \frac{\partial (V_t + d_t)}{\partial K_{t+1}} = -1 + \left( \frac{1}{1 + r} \right) [A_{t+1}F_K (K_{t+1}, L_{t+1}) + 1] = 0
  \]

  \[A_{t+1}F_K (K_{t+1}, L_{t+1}) = r\]

  \item FO condition with respect to $L_s$

  \[
  \frac{\partial (V_t + d_t)}{\partial L_t} = A_tF_L (K_t, L_t) - w_t = 0
  \]
\end{itemize}
4.5 Alternative Intertemporal Budget Constraint

- CRS implies

$$AF(K, L) = AF_K K + AF_L L = rK + wL$$

- Using this in the value of stock expression

$$V_t = \sum_{s=t+1}^{\infty} \left( \frac{1}{1 + r} \right)^{s-t} [rK_s - (K_{s+1} - K_s)]$$

$$= \sum_{s=t+1}^{\infty} \left( \frac{1}{1 + r} \right)^{s-t} [(1 + r)K_s - K_{s+1}]$$

$$= \left( \frac{1}{1 + r} \right) [(1 + r)K_{t+1} - K_{t+2}] + \left( \frac{1}{1 + r} \right)^2 [(1 + r)K_{t+2} - K_{t+3}] + \ldots$$

$$= K_{t+1} - \lim_{T \to \infty} \left( \frac{1}{1 + r} \right)^{t+T} K_{t+T+1}$$
- Impose limit equal to zero

- Ex dividend market value of the firm is the value of capital in place for production next period

- Country’s financial wealth is the sum of its net foreign assets plus capital

\[ Q = B + K \]

- IBC can be written

\[
\sum_{s=t}^{\infty} C_s \left( \frac{1}{1 + r} \right)^{t-s} = (1 + r) Q_t + \sum_{s=t}^{\infty} (w_s L - G_s) \left( \frac{1}{1 + r} \right)^{t-s}
\]
- When $\beta (1 + r) = 1$

$$C_t = rQ_t + \frac{r}{1+r} \sum_{s=t}^{\infty} (w_s L - G_s) \left( \frac{1}{1+r} \right)^{t-s}$$

$$= rQ_t + \tilde{w}_t L - \tilde{G}_t$$

- Alternative current account expression

$$S_t = rQ_t + w_t L - G_t - C_t$$

$$CA_t = S_t - I_t = w_t L - \tilde{w}_t L - (G_t - \tilde{G}) - I_t$$

* Positive future productivity shock raises $I_t$ for one period and $\tilde{w}_t$ permanently

* CA deficit for one period

* Deficit could last longer if investment takes more than one period