1. Assume that an open economy is populated by representative agents with utility given by:

$$U = \sum_{t=0}^{\infty} \beta^t \left( \gamma \ln C_t + (1 - \gamma) \ln \left( \frac{M_t}{P_t} \right) \right),$$

where $\beta < 1$ represents the discount factor, $C$ represents real consumption, $M$ is nominal money balances, and $P$ is the price level. The agent receives exogenous real endowment income ($Y$) each period and pays taxes ($\tau$). He has access to an international real bond ($B$). His flow budget constraint is given by:

$$B_{t+1} + \frac{M_t}{P_t} = (1 + r) B_t + \frac{M_{t-1}}{P_{t-1}} + Y_t - C_t - \tau_t.$$

He also must satisfy:

$$\lim_{t \to \infty} \left( B_{t+1} + \frac{M_t}{P_t} \right) \left( \frac{1}{1 + r} \right)^t = 0.$$

(a) Compute the first order conditions and solve for an expression for expenditures on money as a function of consumption expenditures. Consider this to be a money demand function and take natural logarithms.

$$\frac{C_{t+1}}{C_t} = \beta (1 + r)$$

$$\gamma \frac{1}{C_t P_t} + (1 - \gamma) \frac{1}{M_t} - \beta \gamma \frac{1}{C_{t+1} P_{t+1}} = 0$$

Substituting yields an expression relating current consumption to real money:

$$\frac{M_t}{P_t} = \left( \frac{1 - \gamma}{\gamma} \right) \frac{C_t}{i_{t+1}/(1 + i_{t+1})}.$$

Letting small letters denote natural logs (with the exception of the interest rate)

$$m_t - p_t = \ln \left( \frac{1 - \gamma}{\gamma} \right) + c_t - \ln \left( \frac{i_{t+1}}{1 + i_{t+1}} \right)$$

(b) You end up with an interest rate expression which does not simplify. Linearize this about $i$, where this represents a long-run equilibrium interest rate. Now, write the money demand function in the logs of money, price, consumption and in the interest rate (not its log).

Linearizing about $i_{t+1} = i$ yields:

$$\ln \left( \frac{i_{t+1}}{1 + i_{t+1}} \right) = \ln \left( \frac{i}{1 + i} \right) + \frac{1}{i(1 + i)} (i_{t+1} - i)$$
Money demand becomes:

$$m_t - p_t = \ln \left( \frac{1 - \gamma}{\gamma} \right) + c_t - \ln \left( \frac{i}{1+i} \right) + \frac{1}{i(1+i)} (i_{t+1} - i).$$

Letting

$$h = \ln \left( \frac{1 - \gamma}{\gamma} \right) - \ln \left( \frac{i}{1+i} \right) - \frac{i}{i(1+i)},$$

and

$$\alpha = \frac{1}{i(1+i)},$$

money demand becomes

$$m_t - p_t = h + c_t - \alpha i_{t+1}.$$

(c) Assume that the foreign country has the same money demand function with the same parameters. They can have different values for variables. Write the foreign money demand function, denoting foreign variables by an asterisk.

$$m_t^* - p_t^* = h + c_t^* - \alpha i_{t+1}^*$$

(d) Impose purchasing power parity and solve for the exchange rate.

Purchasing power parity requires

$$p_t - p_t^* = e_t.$$

Subtracting one money demand equation from the other, solving for the relative price, and imposing PPP yields:

$$e_t = p_t - p_t^* = m_t - m_t^* - (c_t - c_t^*) + \alpha (i_{t+1} - i_{t+1}^*)$$

(e) Write the expression for interest rate parity. Take logs and write an approximation for interest rate parity in terms of interest rates and log exchange rates.

Interest rate parity requires:

$$1 + i_{t+1} = \frac{E_{t+1}}{E_t} (1 + i_{t+1}^*)$$

Taking logs yields

$$i_{t+1} - i_{t+1}^* \approx e_{t+1} - e_t.$$

(f) Substitute the approximation for interest rate parity into the expression for the exchange rate and solve for the exchange rate as a function of the future exchange rate.

$$e_t = \frac{m_t - m_t^* - (c_t - c_t^*) + \alpha e_{t+1}}{1 + \alpha}.$$
(g) Now, solve this equation forward to get the exchange rate as a function of expected future relative money and relative consumption. Use this equation to explain what causes sustained exchange rate depreciation. Compare this to the causes of sustained inflation.

Let

\[ f_t = m_t - m^*_t - (c_t - c^*_t) \]

\[ e_t = \sum_{s=0}^{\infty} \frac{1}{1 + \alpha} \left[ \frac{\alpha}{1 + \alpha} \right]^s f_{t+s} + \lim_{T \to \infty} \left[ \frac{\alpha}{1 + \alpha} \right]^T e_{t+T}, \]

where the no bubbles solution imposes a zero limit term. Relative money growth in excess of consumption growth (generally caused by output growth) creates sustained exchange rate depreciation. Money growth in excess of output growth creates sustained inflation.

(h) Now, return to the equation in which the exchange rate is expressed as a function of relative money, relative consumption, and relative interest rates. Assume that \( \beta (1 + r) = 1 \) so that consumption is constant. Assume that the country pegs the exchange rate. What does this imply about the interest rate differential? Now, let the log of the peg be \( \bar{e} \). Solve for the value of \( \ln M \) which will assure this pegged rate is an equilibrium.

With a pegged exchange rate, the exchange rate cannot be changing so interest rate parity implies that the interest rate differential must be zero.

\[ m_t = \bar{e} + \bar{e} - \bar{e}^* + m^*_t. \]

Note that the foreign country controls the money supply for both countries when the domestic country decides to peg the exchange rate. For what follows, assume that the foreign country fixes the money supply.

(i) Let:

\[ M_t = D_t + R_t, \]

where \( D \) is domestic credit and \( R \) is reserves. Assume that domestic credit is growing at rate \( g \) such that

\[ D_t = D_0 e^{gt}. \]

Solve for the rate of growth of reserves under the pegged rate regime.

\[ D_0 e^{gt} + R_t = M \]

Take the time derivative of both sides of the equation:

\[ gD_0 e^{gt} + \frac{dR}{dt} = 0. \]

(j) Define the shadow exchange rate as the equilibrium exchange rate once reserves have been exhausted. Write an equation for the logarithm of the shadow exchange rate.

\[ s_t = d_t - m^* - \bar{e} + \bar{e}^* + \alpha g \]

where \( g \) is the rate of change of the exchange rate after the collapse of the fixed rate, or equivalently the interest rate differential.
(k) What is the criterion for determining the date of exchange rate collapse? Compute
collapse time, $T$. Explain intuitively its determinants.
The fixed exchange rate regime will end on the date at which the shadow exchange
rate equals the fixed exchange rate. The equation for the fixed exchange rate can
be expressed as
$$
\bar{e} = m_0 - m^* - \bar{c} + \bar{c}^*
$$
Subtracting the equation for the for the fixed rate from the equation for the
shadow rate yields
$$
s_t - \bar{e} = \ln D_0 e^{gt} - m_0 + \alpha g
$$
The collapse time $T$ is the value of $t$ which sets the difference between the shadow
rate and the fixed rate equal to zero.
$$
T = \frac{m_0 - d_0 - \alpha g}{g}
$$

(l) Draw graphs for time paths of the log exchange rate, log shadow rate, log money
and log reserves. Explain the time paths.
The exchange rate is fixed until time $T$, at which point it grows at the rate $g$. Money is fixed until date $T$. On date $T$, if falls discretely, and then begins
rising at rate $g$. The shadow rate rises continuously as $d$ grows, and reserves fall
continuously until date $T$, when they are eliminated.

(m) Explain intuitively why a speculative attack is a necessary component of equi-
librium. A speculative attack is necessary to yield post-collapse money market
equilibrium. Either money or the exchange rate must change discretely because
post-collapse money demand is lower. The exchange rate cannot change discretely
in equilibrium. Therefore, the money supply must fall with the attack.