Stochastic Calculus

Financial Economics
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Question 1

Do the following using the Itô stochastic calculus. Let t and z denote time and a Brownian motion, respectively.

a) Suppose

$$\mathrm{d}y = a\,\mathrm{d}t + b\,\mathrm{d}z.$$

Show whether $\mathrm{d}(\ln y) = (\mathrm{d}y)/y$.

b) Suppose

$$\begin{aligned} \mathrm{d}x &= 3x\,\mathrm{d}z \\ \mathrm{d}y &= 2y\,\mathrm{d}t. \end{aligned}$$

If $w = x^2 y$, calculate $\mathrm{d}w$.

Answer 1

a) Evaluate the second-order Taylor series:

$$\mathrm{d}\ln y = \frac{\mathrm{d}\ln y}{\mathrm{d}y}\mathrm{d}y + \frac{1}{2}\frac{\mathrm{d}^2 \ln y}{\mathrm{d}y^2}(\mathrm{d}y)^2 = \frac{\mathrm{d}y}{y} + \frac{1}{2}\left(\frac{-1}{y^2}\right)(b^2\,\mathrm{d}t),$$

so

$$\mathrm{d}\ln y \neq \frac{\mathrm{d}y}{y}.$$

b) Evaluate the second-order Taylor series:

$$\begin{aligned} \mathrm{d}w &= \frac{\partial w}{\partial x}\mathrm{d}x + \frac{\partial w}{\partial y}\mathrm{d}y + \frac{1}{2}\frac{\partial^2 w}{\partial x^2}(\mathrm{d}x)^2 + \frac{\partial^2 w}{\partial x \partial y}\mathrm{d}x\,\mathrm{d}y + \frac{1}{2}\frac{\partial^2 w}{\partial y^2}(\mathrm{d}y)^2 \\ &= (2xy)(3x\,\mathrm{d}z) + (x^2)(2y\,\mathrm{d}t) + \frac{1}{2}(2y)(9x^2\,\mathrm{d}t) + (2x)(0) + \frac{1}{2}(0)(0) \\ &= 11x^2 y\,\mathrm{d}t + 6x^2 y\,\mathrm{d}z, \end{aligned}$$

so

$$\mathrm{d}w = 11w\,\mathrm{d}t + 6w\,\mathrm{d}z.$$

Question 2

Prove the following rule of stochastic calculus:

$$\mathrm{d}t\,\mathrm{d}z = 0$$

(t is time, and z is Brownian motion). To prove the rule, let the time interval $\Delta t \to 0$ in a discrete-time model.

Answer 2

Define

$$\Delta z_i \equiv z(i\Delta t) - z[(i-1)\Delta t],$$

where $i = 1, \ldots, n$, such that $n\Delta t = t$.

We have

$$\begin{aligned} \int_0^t \mathrm{d}t\,\mathrm{d}z &= \lim_{\Delta t \to 0} \sum_{i=1}^n \Delta t \Delta z_i \\ &= \lim_{\Delta t \to 0} \Delta t \sum_{i=1}^n \Delta z_i \\ &= \lim_{\Delta t \to 0} \Delta t [z(t) - z(0)] \\ &= 0. \end{aligned}$$

Alternatively,

$$\begin{aligned} \int_0^t dt\,dz &= \lim_{\Delta t \to 0} \Delta t \sum_{i=1}^n \Delta z_i \\ &= \lim_{n \to \infty} t \left(\frac{1}{n}\sum_{i=1}^n \Delta z_i\right) \\ &= 0; \end{aligned}$$

by the law of large numbers, the sample mean of the Δz_i converges to 0.

Question 3

Suppose that $\mathrm{d}p$ and $\mathrm{d}q$ are Itô stochastic differentials.

a) Prove the product rule $\mathrm{d}(pq) = q\,\mathrm{d}p + p\,\mathrm{d}q + \mathrm{d}p\,\mathrm{d}q$.

b) Consider the special case $\mathrm{d}p = 3p\,\mathrm{d}t + 4q\,\mathrm{d}z$ and $\mathrm{d}q = 2pq\,\mathrm{d}t + 3q\,\mathrm{d}z$ (z is Brownian motion). Evaluate $\mathrm{d}(pq)$ in terms of $\mathrm{d}t$ and $\mathrm{d}z$.

Answer 3

a) The second-order Taylor series is

$$\begin{aligned} \mathrm{d}(pq) &= \frac{\partial(pq)}{\partial p}\mathrm{d}p + \frac{\partial(pq)}{\partial q}\mathrm{d}q \\ &\quad + \frac{1}{2}\frac{\partial^2(pq)}{\partial p^2}(\mathrm{d}p)^2 + \frac{1}{2}\frac{\partial^2(pq)}{\partial q^2}(\mathrm{d}q)^2 + \frac{\partial^2(pq)}{\partial p \partial q}\mathrm{d}p\,\mathrm{d}q \\ &= q\,\mathrm{d}p + p\,\mathrm{d}q + 0\,(\mathrm{d}p)^2 + 0\,(\mathrm{d}q)^2 + \mathrm{d}p\,\mathrm{d}q. \end{aligned}$$

b)

$$\begin{aligned} \mathrm{d}(pq) &= q\,\mathrm{d}p + p\,\mathrm{d}q + \mathrm{d}p\,\mathrm{d}q \\ &= q(3p\,\mathrm{d}t + 4q\,\mathrm{d}z) + p(2pq\,\mathrm{d}t + 3q\,\mathrm{d}z) \\ &\quad + (3p\,\mathrm{d}t + 4q\,\mathrm{d}z)(2pq\,\mathrm{d}t + 3q\,\mathrm{d}z) \\ &= (3pq + 2p^2 q + 12q^2)\,\mathrm{d}t + (4q^2 + 3pq)\,\mathrm{d}z. \end{aligned}$$

Question 4

Let P denote the stock price, let D denote the dividend, and let R denote the market interest rate. Suppose that the stock price P follows the Itô stochastic differential equation

$$\mathrm{d}P = \mathrm{E}_t(\mathrm{d}P) + \mathrm{d}z.$$

Market equilibrium requires that the expected capital gain plus the dividend equals the market interest rate times the price,

$$\mathrm{E}_t(\mathrm{d}P) + D\,\mathrm{d}t = RP\,\mathrm{d}t.$$

Assume that the dividend follows the Itô stochastic differential equation $\frac{\mathrm{d}D}{D} = G\,\mathrm{d}t$ (no instantaneous stochastic part).

Define the price/dividend ratio $\theta = P/D$. Using Itô's formula, show that the equilibrium condition is equivalent to the following:

$$\mathrm{E}_t(\mathrm{d}\theta) = [-1 + (R - G)\theta]\,\mathrm{d}t.$$

Explain what happens to the second-order terms.

Answer 4

The second-order Taylor series for $\mathrm{d}\theta$ is

$$\mathrm{d}\theta = \frac{\partial\theta}{\partial P}\mathrm{d}P + \frac{\partial\theta}{\partial D}\mathrm{d}D + + \frac{1}{2}\frac{\partial^2\theta}{\partial P^2}(\mathrm{d}P)^2 + \frac{1}{2}\frac{\partial^2\theta}{\partial D^2}(\mathrm{d}D)^2 + \frac{\partial^2\theta}{\partial P \partial D}\mathrm{d}P\,\mathrm{d}D.$$

The second-order terms are all zero. The second derivative $\frac{\partial^2\theta}{\partial P^2} = 0$. The products $(\mathrm{d}D)^2 = 0$ and $\mathrm{d}P\,\mathrm{d}D = 0$, since $\mathrm{d}D$ contains no instantaneous stochastic part. The remaining first-order terms reduce to

$$\mathrm{d}\theta = \theta\left(\frac{\mathrm{d}P}{P} - \frac{\mathrm{d}D}{D}\right).$$

Taking the expected value, we have

$$\begin{aligned} \mathrm{E}_t(\mathrm{d}\theta) &= \theta\left(\frac{\mathrm{E}_t(\mathrm{d}P)}{P} - \frac{\mathrm{E}_t(\mathrm{d}D)}{D}\right) \\ &= \theta\left[\left(R - \frac{1}{\theta}\right) - G\right]\mathrm{d}t \\ &= [-1 + (R - G)\theta]\,\mathrm{d}t. \end{aligned}$$

Question 5

Consider the following two stochastic differential equations:

$$\mathrm{d}\ln x = \mathrm{d}z$$

and

$$\frac{\mathrm{d}x}{x} = \mathrm{d}z,$$

both with initial condition $x_0 = 1$.

a) Solve both equations exactly for x as a function of z.

b) Approximate both equations as stochastic difference equations:

$$\begin{aligned} \ln x_{t+\Delta t} - \ln x_t &= \Delta z_t \\ \frac{x_{t+\Delta t} - x_t}{x_t} &= \Delta z_t. \end{aligned}$$

Here z_t is a discrete-time random walk; the first difference Δz_t is white noise, distributed $N(0, \Delta t)$. Solve these difference equations as a simulation, using the same realization of z for both, derived from a random number generator. Choose a very short time period Δt, and solve for $0 \leq t \leq 1$. Plot the results, and show that the computations support your exact solution in part (a).

Answer 5

Consider first the stochastic differential equation $\mathrm{d}\ln x = \mathrm{d}z$. Integrating it gives

$$\int_0^t \mathrm{d}\ln x = \int_0^t \mathrm{d}z = z(t) - z(0) = z(t),$$

since $z(0) = 0$. Of course

$$\int_0^t \mathrm{d}\ln x = \ln x(t) - \ln x(0) = \ln x(t).$$

Hence

$$\ln x(t) = z(t).$$

Alternatively, consider the stochastic differential equation $\mathrm{d}x/x = \mathrm{d}z$. Taking the second-order Taylor series for $\mathrm{d}\ln x$, we have

$$\begin{aligned} \mathrm{d}\ln x &= \frac{\mathrm{d}\ln x}{\mathrm{d}x}\mathrm{d}x + \frac{1}{2}\frac{\mathrm{d}^2 \ln x}{\mathrm{d}x^2}(\mathrm{d}x)^2 \\ &= \left(\frac{1}{x}\right)\mathrm{d}x + \frac{1}{2}\left(\frac{-1}{x^2}\right)(\mathrm{d}x)^2 \\ &= \frac{\mathrm{d}x}{x} - \frac{1}{2}\left(\frac{\mathrm{d}x}{x}\right)^2 \\ &= \mathrm{d}z - \frac{1}{2}(\mathrm{d}z)^2 \\ &= \mathrm{d}z - \frac{1}{2}\mathrm{d}t. \end{aligned}$$

Integrating this expression gives

$$\ln x(t) = z(t) - \frac{1}{2}t.$$

Thus the two exact solutions differ by $\frac{1}{2}t$. The idea is to compare the two approximate solutions for $\ln x(t)$ computed from the two difference equations. They should differ by approximately $\frac{1}{2}t$, as long as the time period Δt is small.