Stochastic Difference

Let Δt denote the length of a time period. For a random variable x_t, define the *stochastic difference*

$$\Delta x_t := x_{t+\Delta t} - x_t;$$

the difference is from the present into the future, not from the past to the present ($\Delta x_t :\neq x_t - x_{t-\Delta t}$).

Of course

$$\begin{aligned} x_t &= x_0 + (x_{\Delta t} - x_0) + (x_{2\Delta t} - x_{\Delta t}) + \cdots \\ &\quad + (x_{t-\Delta t} - x_{t-2\Delta t}) + (x_t - x_{t-\Delta t}) \\ &= x_0 + \Delta x_0 + \Delta x_{\Delta t} + \cdots + \Delta x_{t-2\Delta t} + \Delta x_{t-\Delta t}. \end{aligned}$$

The value at time t is the initial value plus the sum of the stochastic differences.

Discrete-Time, Normal Random Walk

Assume

$$\begin{aligned} \Delta x_t &\sim \mathrm{N}(0, \Delta t) \\ n\Delta t &= t, \end{aligned}$$

such that Δx_t is uncorrelated for different periods (*white noise*). For $x_0 = 0$, then

$$x_t \sim \mathrm{N}(0, t).$$

We refer to x_t as a *random walk*.

Wiener-Brownian Motion

Intuitively, Wiener-Brownian motion is the continuous-time limit of the discrete-time, normal random walk, as $\Delta t \to 0$:

$$x_t = x_0 + \lim_{\Delta t \to 0} (\Delta x_0 + \Delta x_{\Delta t} + \cdots + \Delta x_{t-2\Delta t} + \Delta x_{t-\Delta t}).$$

Stochastic Differential

Although time is continuous, it is useful to think of the analysis as dealing with an infinitesimal time period of length $\mathrm{d}t$. The current time is t and next period is $t + \mathrm{d}t$. Define the stochastic differential $\mathrm{d}x_t := x_{t+\mathrm{d}t} - x_t$; the difference is from the present into the future, not from the past to the present ($\mathrm{d}x_t :\neq x_t - x_{t-\mathrm{d}t}$).

In the limit, the sum of the changes in the stochastic differences is the integral of the stochastic differential,

$$\begin{aligned} x_t - x_0 &= \lim_{\Delta t \to 0} (\Delta x_0 + \Delta x_{\Delta t} + \cdots + \Delta x_{t-2\Delta t} + \Delta x_{t-\Delta t}) \\ &= \int_0^t \mathrm{d}x_t. \end{aligned}$$

Wiener-Brownian Motion

In the limit,

$$\mathrm{d}x_t \sim \mathrm{N}(0, \mathrm{d}t),$$

in which $\mathrm{d}x_t$ is white noise. For $x_0 = 0$, then

$$x_t = \mathrm{N}(0, t).$$

We refer to this stochastic process as *Wiener-Brownian motion*.

Properties of Wiener-Brownian Motion

Almost everywhere, x_t is

- Continuous;
- Non-differentiable.

Differentiability would imply that the change in x_t can be forecasted, which would contradict the white noise property.

Using measure theory, Wiener proved these results rigorously.