Notation

Let P_t^τ denote the price at time t of a risk-free, pure-discount bond worth one dollar at its maturity in τ years, at time $t+\tau$. Thus $P_t^0 = 1$.

Let R_t^τ denote the yield to maturity on this bond. By definition,

$$P_t^\tau = \mathrm{e}^{-\tau R_t^\tau}.$$

Let us assume that the prices and the interest rates follow a stationary stochastic process.

Expectations Theory

The expectations theory says that the long-term interest rate is the average of current and expected future short-term rates. For example, the two-year interest rate is the average of the current one-year rate and the one-year rate expected for next year,

$$R_t^2 = \frac{1}{2}\left[R_t^1 + \mathrm{E}_t\left(R_{t+1}^1\right)\right]. \tag{1}$$

Variance Bounds

Shiller [1] shows how the expectations theory implies that the variability of the short-term interest rate sets an upper bound on the variability of the long-term interest rate, and he studies whether United States data satisfy these variance bounds.

We work out the bounds in the context of the one-year and two-year pure-discount bonds. Shiller's analysis is analogous but more complex (coupon-bearing, long-term bonds).

We assume that expectations are rational.

Forecast Error

Define the forecast error

$$e_t := R_t^1 - \mathrm{E}_{t-1}\left(R_t^1\right).$$

Since the forecast error is uncorrelated with the forecast,

$$\mathrm{Var}\left(R_t^1\right) = \mathrm{Var}\left[\mathrm{E}_{t-1}\left(R_t^1\right)\right] + \mathrm{Var}\left(e_t\right).$$

Hence

$$\mathrm{Var}\left(e_t\right) \leq \mathrm{Var}\left(R_t^1\right). \tag{2}$$

Also, since R_t^1 is a possible forecast of R_{t+1}^1, with forecast error $\mathrm{Var}\left(\Delta R_t^1\right)$,

$$\mathrm{Var}\left(e_t\right) \leq \mathrm{Var}\left(\Delta R_t^1\right). \tag{3}$$

Ex Post Rational Long-Term Interest Rate

The *ex post* rational long-term interest rate is

$$R_t^{2*} := \frac{1}{2}\left(R_t^1 + R_{t+1}^1\right). \tag{4}$$

Under perfect foresight, the long-term interest rate would equal this value.

With uncertainty, however,

$$R_t^2 = \mathrm{E}_t\left(R_t^{2*}\right).$$

We have

$$\begin{aligned} R_t^{2*} &= \frac{1}{2}\left(R_t^1 + R_{t+1}^1\right) \\ &= \frac{1}{2}\left[R_t^1 + \mathrm{E}_t\left(R_{t+1}^1\right)\right] + \frac{1}{2}\left[R_{t+1}^1 - \mathrm{E}_t\left(R_{t+1}^1\right)\right] \\ &= R_t^2 + e_{t+1}, \end{aligned}$$

by (1).

Variance Bound

A variance bound is

$$\mathrm{Var}\left(R_t^{2*}\right) \geq \mathrm{Var}\left(R_t^2\right).$$

This inequality is comparable to [1, p. 1202], which is violated by the data: the variance of the *ex post* rational interest rate is low, and the variance of the long-term interest rate is higher.

Holding-Period Return

Define the one-period holding-period return on a two-year bond,

$$H_{t+1}^{2} := \ln P_{t+1}^{1} - \ln P_{t}^{2}.$$

Hence

$$\begin{aligned} H^2_{t+1} &= -R^1_{t+1} + 2R^2_t \\ &= -R^1_{t+1} + \left[R^1_t + \mathrm{E}_t\left(R^1_{t+1}\right)\right] \text{ by (1)} \\ &= R^1_t - e_{t+1}. \end{aligned}$$

Of course this relationship expresses the basis of the expectations theory: all bonds have the same expected holding-period return.

The expected value $\mathrm{E}_t\left(H^2_{t+1}\right) = R^1_t$, and the error is $-e_{t+1}$, so

$$\mathrm{Var}\left(H^2_{t+1}\right) = \mathrm{Var}\left(R^1_t\right) + \mathrm{Var}\left(e_{t+1}\right),$$

since the error is uncorrelated with the forecast. Therefore

$$\mathrm{Var}\left(R^1_t\right) \leq \mathrm{Var}\left(H^2_t\right) \leq 2\mathrm{Var}\left(R^1_t\right),$$

by (2). The right inequality is comparable to [1, I.1, p. 1203], which is violated by the data: the holding-period return on the long-term bond fluctuates too much relative to the short-term interest rate.

Excess Holding-Period Return

Since the excess holding-period return on the two-year bond is

$$H_{t+1}^{2} - R_{t}^{1} = -e_{t+1}.$$

therefore

$$\operatorname{Var}\left(H_{t+1}^{2} - R_{t}^{1}\right) \leq \operatorname{Var}\left(\Delta R_{t}^{1}\right),$$

by (3). This inequality corresponds to [1, I.2, p. 1204], which is satisfied by the data.

Averaging

That the *ex post* rational interest rate is the average of the current and future short-term rates obtains another inequality:

$$\begin{aligned}\mathrm{Var}\left(R_t^{2*}\right) &= \mathrm{Var}\left[\frac{1}{2}\left(R_t^1 + R_{t+1}^1\right)\right] \\ &= \frac{1+\rho}{2}\mathrm{Var}\left(R_t^1\right) \\ &\leq \mathrm{Var}\left(R_t^1\right).\end{aligned}$$

Here ρ is the first-order autocorrelation of R_t^1.

This inequality corresponds to [1, p. 1202]. Unlike the inequalities above, this inequality depends only on stationarity, but not on either the expectations theory or rational expectations. It is supported by the data.

References

[1] Robert J. Shiller. The volatility of long-term interest rates and expectations models of the term stucture. *Journal of Political Economy*, 87(6):1190–1219, December 1979. HB1J7.