Notation

Let $P_t^\tau$ denote the price at time $t$ of a risk-free, pure-discount bond worth one dollar at its maturity in $\tau$ years, at time $t + \tau$. Thus $P_t^0 = 1$.

Let $R_t^\tau$ denote the yield to maturity on this bond. By definition,

$$P_t^\tau = e^{-\tau R_t^\tau}.$$

Let us assume that the prices and the interest rates follow a stationary stochastic process.
Expectations Theory

The expectations theory says that the long-term interest rate is the average of current and expected future short-term rates. For example, the two-year interest rate is the average of the current one-year rate and the one-year rate expected for next year,

\[ R^2_t = \frac{1}{2} \left[ R^1_t + E_t \left( R^1_{t+1} \right) \right]. \]  \hfill (1)
Variance Bounds

Shiller [1] shows how the expectations theory implies that the variability of the short-term interest rate sets an upper bound on the variability of the long-term interest rate, and he studies whether United States data satisfy these variance bounds.

We work out the bounds in the context of the one-year and two-year pure-discount bonds. Shiller’s analysis is analogous but more complex (coupon-bearing, long-term bonds).

We assume that expectations are rational.
Define the forecast error

\[ e_t := R_t^1 - E_{t-1} (R_t^1) . \]

Since the forecast error is uncorrelated with the forecast,

\[ \text{Var} (R_t^1) = \text{Var} [E_{t-1} (R_t^1)] + \text{Var} (e_t) . \]

Hence

\[ \text{Var} (e_t) \leq \text{Var} (R_t^1) . \quad (2) \]

Also, since \( R_t^1 \) is a possible forecast of \( R_{t+1}^1 \), with forecast error \( \text{Var} (\Delta R_t^1) \),

\[ \text{Var} (e_t) \leq \text{Var} (\Delta R_t^1) . \quad (3) \]


**Ex Post Rational Long-Term Interest Rate**

The *ex post* rational long-term interest rate is

$$ R_t^{2*} := \frac{1}{2} \left( R_t^1 + R_{t+1}^1 \right) . $$

(4)

Under perfect foresight, the long-term interest rate would equal this value.

With uncertainty, however,

$$ R_t^2 = E_t \left( R_t^{2*} \right) . $$
We have

\[ R_{t}^{2*} = \frac{1}{2} (R_{t}^{1} + R_{t+1}^{1}) \]

\[ = \frac{1}{2} \left[ R_{t}^{1} + E_t \left( R_{t+1}^{1} \right) \right] + \frac{1}{2} \left[ R_{t+1}^{1} - E_t \left( R_{t+1}^{1} \right) \right] \]

\[ = R_{t}^{2} + e_{t+1}, \]

by (1).
Variance Bound

A variance bound is

\[ \text{Var} (R_t^2) \geq \text{Var} (R_t^2). \]

This inequality is comparable to [1, p. 1202], which is violated by the data: the variance of the \textit{ex post} rational interest rate is low, and the variance of the long-term interest rate is higher.
Holding-Period Return

Define the one-period holding-period return on a two-year bond,

\[ H_{t+1}^2 := \ln P_{t+1}^1 - \ln P_t^2. \]
Hence

\[ H_{t+1}^2 = -R_{t+1}^1 + 2R_t^2 \]
\[ = -R_{t+1}^1 + \left[ R_t^1 + E_t \left( R_{t+1}^1 \right) \right] \text{ by (1)} \]
\[ = R_t^1 - e_{t+1}. \]

Of course this relationship expresses the basis of the expectations theory: all bonds have the same expected holding-period return.
The expected value $E_t \left( H_{t+1}^2 \right) = R_t^1$, and the error is $-e_{t+1}$, so

$$\text{Var} \left( H_{t+1}^2 \right) = \text{Var} \left( R_t^1 \right) + \text{Var} \left( e_{t+1} \right),$$

since the error is uncorrelated with the forecast. Therefore

$$\text{Var} \left( R_t^1 \right) \leq \text{Var} \left( H_t^2 \right) \leq 2\text{Var} \left( R_t^1 \right),$$

by (2). The right inequality is comparable to [1, I.1, p. 1203], which is violated by the data: the holding-period return on the long-term bond fluctuates too much relative to the short-term interest rate.
Excess Holding-Period Return

Since the excess holding-period return on the two-year bond is

\[ H_{t+1}^2 - R_t^1 = -e_{t+1}. \]

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therefore

\[ \text{Var} \left( H_{t+1}^2 - R_t^1 \right) \leq \text{Var} \left( \Delta R_t^1 \right), \]

by (3). This inequality corresponds to [1, I.2, p. 1204], which is satisfied by the data.
Averaging

That the *ex post* rational interest rate is the average of the current and future short-term rates obtains another inequality:

\[
\text{Var} (R_t^{2*}) = \text{Var} \left[ \frac{1}{2} (R_t^1 + R_{t+1}^1) \right] \\
= \frac{1 + \rho}{2} \text{Var} (R_t^1) \\
\leq \text{Var} (R_t^1).
\]

Here \( \rho \) is the first-order autocorrelation of \( R_t^1 \).
This inequality corresponds to [1, p. 1202]. Unlike the inequalities above, this inequality depends only on stationarity, but not on either the expectations theory or rational expectations. It is supported by the data.
References