Erratic Price, Smooth Dividend

Shiller [1] argues that the stock market is inefficient: stock prices fluctuate too much.

According to economic theory, the stock price should equal the present value of expected dividends. However dividends are very stable; they fluctuate very little about an upward trend. Expected dividends should therefore also fluctuate little, and consequently stock prices should be stable.

In fact, stock prices fluctuate wildly.

Variance Bound

Shiller shows how the variability of the dividend sets an upper bound to the variability of the stock price.

Let $p_t$ denote the stock price at time $t$, and let $d_t$ denote the dividend at time $t$. Let us assume that $p_t$ and $d_t$ follow a stationary Itô stochastic process.

Present Value

According to economic theory, the price is the present value of expected dividends,

$$ p_t = \int_t^\infty E_t (d_\tau) e^{-r(\tau-t)} \, d\tau. \tag{1} $$

The equivalent rate-of-return condition is

$$ d_t \, dt + E_t (dp_t) = rp_t \, dt, $$

the dividend plus the capital gain equals the market interest rate times the price.

Ex Post Rational Price

Define the “ex post rational price” $p_t^*$ as the present value of actual dividends,

$$ p_t^* = \int_t^\infty d_\tau e^{-r(\tau-t)} \, d\tau. \tag{2} $$

Economic theory distinguishes between ex post—meaning “after”—and ex ante—meaning “before.” The ex ante (rational) price is just the actual price, based on the present value of expected dividends.

Standard and Poor’s Composite Stock-Price Index

In figure 1 [1, p. 422], the price is the Standard and Poor’s Composite Stock-Price Index for 1871-1979, detrended by an exponential growth factor. The ex post rational price is calculated from the dividend.

Whereas the ex post rational price is stable, the actual price is erratic.
Forecast

By (1) and (2), the price is the forecast of the *ex post* rational price,

\[ p_t = E(p_t^*) . \]

We write

\[ p_t^* = p_t + e_t , \quad (3) \]

in which \( e_t \) denotes the forecast error.

Variance Bound

The inequality

\[ \text{Var}(p_t^*) \geq \text{Var}(p_t) \]

is a *variance bound*. The variance of the *ex post* rational price is an upper limit (upper bound) on the variance of the price.

Moving Average

Since

\[ de_t = dp_t^* - dp_t = re_t \, dt - \sigma \, dz_t , \]

therefore

\[ e_t = \int_t^\infty e^{-r(t-\tau)} \sigma \, dz_{\tau} . \quad (5) \]

The forecast error depends on the future unexpected changes in the price. The forecast error is a moving average of these unexpected changes and exhibits considerable autocorrelation. This autocorrelation is evident in figure 1.

Variance Decomposition

If the forecast (3) is rational, then \( p_t \) and \( e_t \) must be uncorrelated. Consequently

\[ \text{Var}(p_t^*) = \text{Var}(p_t) + \text{Var}(e_t) . \quad (4) \]

Forecast Variance

We calculate the difference between the two variances, by finding the variance of the forecast error.

The price follows the stochastic differential equation

\[ dp_t = (r p_t - d_t) \, dt + \sigma \, dz_t , \]

in accord with the rate-of-return condition. The *ex post* rational price satisfies

\[ dp_t^* = (r p_t^* - d_t) \, dt . \]

Forecast Error Variance

For uncorrelated random variables, the variance of the sum is the sum of the variances. Analogously, the moving average (5) expresses the forecast error as an integral of uncorrelated terms. Hence the variance of the forecast error formula is the integral of the variance of these terms,

\[ \text{Var}(e_t) = \int_0^\infty e^{-2rt} \sigma^2 \, dt = \frac{\sigma^2}{2r} . \]
**Variance-Bound Test**

Unfortunately the variance bound is violated. Figure 1 shows that for detrended data the variance of the price is much larger than the variance of the ex post rational price. Hence the stock market is inefficient.

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**Dividend Yield**

This inefficiency means that one can forecast the rate-of-return on stocks from the dividend yield (the dividend/price ratio).

A profitable trading rule is to buy when the dividend yield is high (because the price is then too low) and to sell when the dividend yield is low (because the price is then too high).

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**Dividend Variance Bound**

The variance of the dividend sets an upper bound to the instantaneous variance of the price.

**Theorem 1 (Dividend Variance Bound)**

\[
\text{Var} (d_t) \geq 2r\sigma^2.
\]  

(6)

Shiller finds that this inequality is also violated, by a large margin: although the dividend is stable, the short-run fluctuation of the price is large.

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Substitute\[E_t (dp_t) = -r [p_t - E(p_t)] \text{d}t.\]

(7)

the continuous-time analogue of a first-order autoregression. Equivalently,

\[E_t (dd_t) = -r [d_t - E(d_t)] \text{d}t,\]

(8)

for the dividend. We show this equivalence below.

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The dividend variance bound (6) holds with equality if and only if

\[E_t (dp_t) = -r [p_t - E(p_t)] \text{d}t.
\]

(7)

the continuous-time analogue of a first-order autoregression. Equivalently,

\[E_t (dd_t) = -r [d_t - E(d_t)] \text{d}t,
\]

(8)

for the dividend. We show this equivalence below.
Taking the stochastic differential obtains
\[ E_t(d d_t) = 2r E_t(d p_t) \]
\[ = 2r \left\{ -r [p_t - E(p_t)] \right\} dr \]
\[ = -2r^2 \left\{ \frac{1}{2r} [d_t - E(d_t)] \right\} dr \]
\[ = -r [d_t - E(d_t)] dt, \]
which proves (8).

A criticism is that the price and the dividend are not stationary, but tend to rise as time passes. The uncertainty about this rate of increase is an important part of the uncertainty. When Shiller detrends the price and the dividend, implicitly he assumes that the long-run rate of increase is known. A change in the expected long-run rate of increase can cause a great change in the price. Consequently the detrending understates the uncertainty about the price and the dividend.

Suppose that a stationary random variable \( x_t \) follows the stochastic differential equation
\[ d x_t = \mu_t dt + \sigma dz_t, \]
where \( z_t \) is Wiener-Brownian motion. Equivalently,
\[ x_{t+dt} - x_t = \mu_t dt + \sigma dz_t. \]
Taking the unconditional expected value gives
\[ 0 = E(d x_t) - E(\mu_t) dt - E(\sigma dz_t) \]
\[ = 0 - E(\mu_t) dt - 0, \]
so the mean \( E(\mu_t) = 0. \)

Taking the unconditional variance gives
\[ V ar(x_{t+dt}) = V ar(x_t + \mu_t dt + \sigma dz_t) \]
\[ = V ar(x_t) + V ar(\mu_t) (dt)^2 + V ar(\sigma dz_t) + 2Cov(x_t, \mu_t) dt + 2Cov(x_t, \sigma dz_t) \]
\[ + 2Cov(\mu_t dt, \sigma dz_t) \]
\[ = V ar(x_t) + \sigma^2 dt + 2Cov(x_t, \mu_t) dt. \]
By stationarity, \( V ar(x_{t+dt}) = V ar(x_t) \), which yields the following lemma.

**Lemma 2 (Covariance of Level and Mean Change)**
\[ Cov(x_t, \mu_t) = -\frac{1}{2} \sigma^2. \]
The covariance of the level and the mean change must be negative.
References