Erratic Price, Smooth Dividend

Shiller [1] argues that the stock market is inefficient: stock prices fluctuate too much.

According to economic theory, the stock price should equal the present value of expected dividends. However dividends are very stable; they fluctuate very little about an upward trend. Expected dividends should therefore also fluctuate little, and consequently stock prices should be stable.

In fact, stock prices fluctuate wildly.

Variance Bound

Shiller shows how the variability of the dividend sets an upper bound to the variability of the stock price.

Let p_t denote the stock price at time t, and let d_t denote the dividend at time t. Let us assume that p_t and d_t follow a stationary Itô stochastic process.

Present Value

According to economic theory, the price is the present value of expected dividends,

$$p_t = \int_t^{\infty} \mathrm{E}_t\,(d_\tau)\,\mathrm{e}^{-r(\tau - t)}\,\mathrm{d}\tau. \tag{1}$$

The equivalent rate-of-return condition is

$$d_t\,\mathrm{d}t + \mathrm{E}_t\,(\mathrm{d}p_t) = r p_t\,\mathrm{d}t,$$

the dividend plus the capital gain equals the market interest rate times the price.

Ex Post Rational Price

Define the "*ex post* rational price" p_t^* as the present value of *actual* dividends,

$$p_t^* := \int_t^\infty d_\tau \mathrm{e}^{-r(\tau-t)}\,\mathrm{d}\tau. \qquad (2)$$

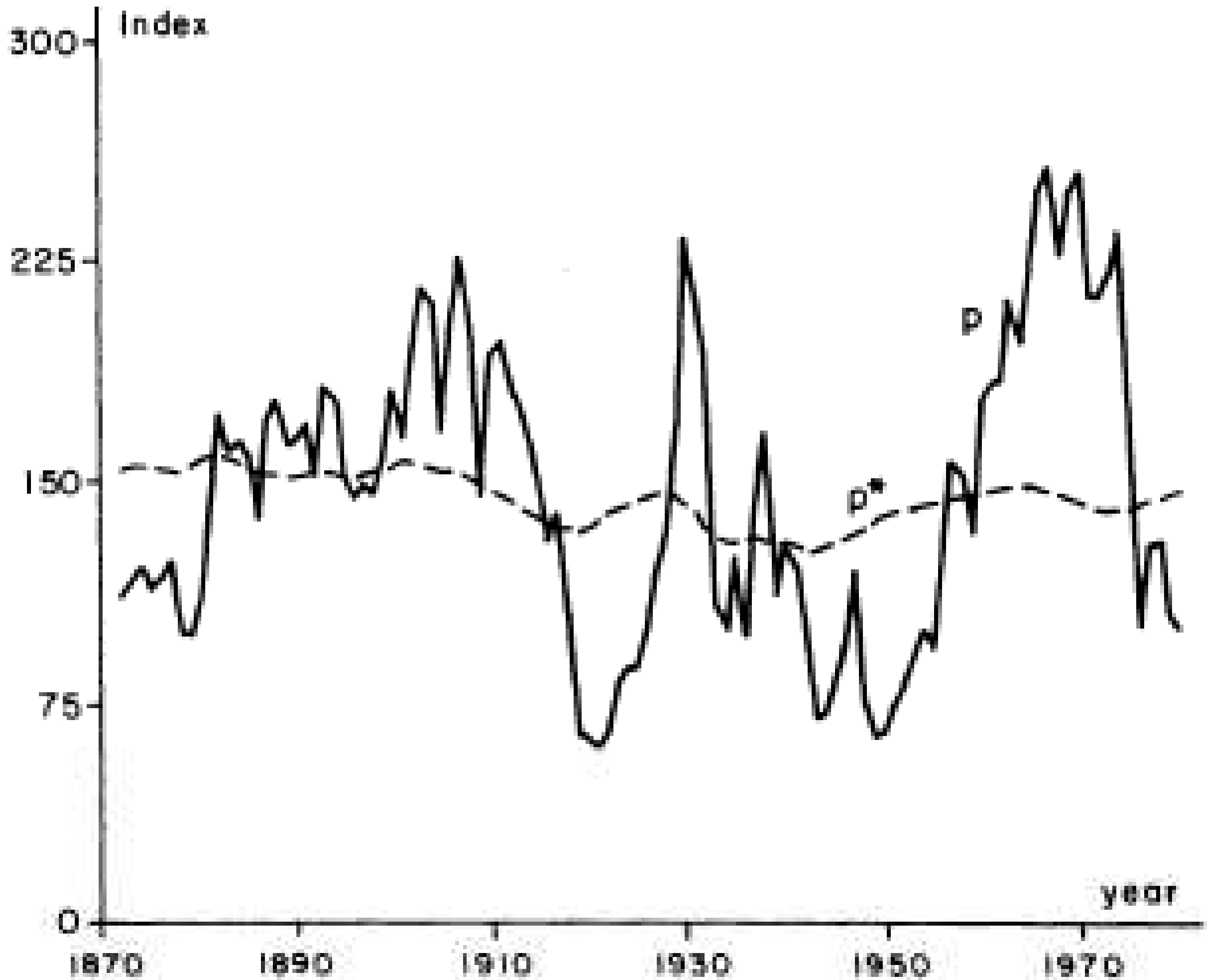
Economic theory distinguishes between *ex post*—meaning "after"—and *ex ante*—meaning "before." The *ex ante* (rational) price is just the actual price, based on the present value of expected dividends.

Standard and Poor's Composite Stock-Price Index

In figure 1 [1, p. 422], the price is the Standard and Poor's Composite Stock-Price Index for 1871-1979, detrended by an exponential growth factor. The *ex post* rational price is calculated from the dividend.

Whereas the *ex post* rational price is stable, the actual price is erratic.

Figure 1: Standard and Poor's Composite



Forecast

By (1) and (2), the price is the forecast of the *ex post* rational price,

$$p_t = \mathrm{E}(p_t^*).$$

We write

$$p_t^* = p_t + e_t, \tag{3}$$

in which e_t denotes the forecast error.

Variance Decomposition

If the forecast (3) is rational, then p_t and e_t must be uncorrelated. Consequently

$$\operatorname{Var}(p_t^*) = \operatorname{Var}(p_t) + \operatorname{Var}(e_t). \tag{4}$$

Variance Bound

The inequality

$$\mathrm{Var}(p_t^*) \geq \mathrm{Var}(p_t)$$

is a *variance bound*. The variance of the *ex post* rational price is an upper limit (upper bound) on the variance of the price.

Forecast Variance

We calculate the difference between the two variances, by finding the variance of the forecast error.

The price follows the stochastic differential equation

$$\mathrm{d}p_t = (rp_t - d_t)\,\mathrm{d}t + \sigma\,\mathrm{d}z_t,$$

in accord with the rate-of-return condition. The *ex post* rational price satisfies

$$\mathrm{d}p_t^* = (rp_t^* - d_t)\,\mathrm{d}t.$$

Moving Average

Since

$$\mathrm{d}e_t = \mathrm{d}p_t^* - \mathrm{d}p_t = re_t\,\mathrm{d}t - \sigma\,\mathrm{d}z_t\,,$$

therefore

$$e_t = \int_t^{\infty} \mathrm{e}^{-r(\tau - t)}\sigma\,\mathrm{d}z_\tau\,. \tag{5}$$

The forecast error depends on the future unexpected changes in the price. The forecast error is a moving average of these unexpected changes and exhibits considerable autocorrelation. This autocorrelation is evident in figure 1.

Forecast Error Variance

For uncorrelated random variables, the variance of the sum is the sum of the variances. Analogously, the moving average (5) expresses the forecast error as an integral of uncorrelated terms. Hence the variance of the forecast error formula is the integral of the variance of these terms,

$$\operatorname{Var}(e_t) = \int_0^\infty \mathrm{e}^{-2rt}\sigma^2\,\mathrm{d}t = \frac{\sigma^2}{2r}.$$

Variance-Bound Test

Unfortunately the variance bound is violated. Figure 1shows that for detrended data the variance of the price is *much* larger than the variance of the *ex post* rational price.

Hence the stock market is inefficient.

Dividend Yield

This inefficiency means that one can forecast the rate-of-return on stocks from the dividend yield (the dividend/price ratio).

A profitable trading rule is to buy when the dividend yield is high (because the price is then too low) and to sell when the dividend yield is low (because the price is then too high).

Dividend Variance Bound

The variance of the dividend sets an upper bound to the instantaneous variance of the price.

Theorem 1 (Dividend Variance Bound)

$$\operatorname{Var}(d_t) \geq 2r\sigma^2. \tag{6}$$

Shiller finds that this inequality is also violated, by a large margin: although the dividend is stable, the short-run fluctuation of the price is large.

The following proves the theorem:

$$
\begin{aligned}
\operatorname{Var}(d_t) &= \operatorname{Var}\left[rp_t - \mathrm{E}_t\left(\mathrm{d}p_t\right)/\mathrm{d}t\right] \\
&\geq -4\operatorname{Cov}\left[rp_t, \mathrm{E}_t\left(\mathrm{d}p_t\right)/\mathrm{d}t\right] \\
&= 2r\sigma^2.
\end{aligned}
$$

The initial equality is the rate-of-return condition. The inequality is an instance of the general relation $\operatorname{Var}(x-y) \geq -4\operatorname{Cov}(x,y)$ (with equality if and only if $\operatorname{Var}(x+y) = 0$). The final equality is lemma 2 below, a general property of a stationary process.

The dividend variance bound (6) holds with equality if and only if

$$\mathrm{E}_t\left(\mathrm{d}p_t\right) = -r\left[p_t - \mathrm{E}\left(p_t\right)\right]\mathrm{d}t. \tag{7}$$

the continuous-time analogue of a first-order autoregression. Equivalently,

$$\mathrm{E}_t\left(\mathrm{d}d_t\right) = -r\left[d_t - \mathrm{E}\left(d_t\right)\right]\mathrm{d}t, \tag{8}$$

for the dividend. We show this equivalence below.

Substitute

$$\mathrm{E}(p_t) = \mathrm{E}(d_t)/r$$

into the rate-of-return condition:

$$\begin{aligned}
\mathrm{E}_t(\mathrm{d}p_t) &= (rp_t - d_t)\,\mathrm{d}t \\
&= \{rp_t - d_t + r[\mathrm{E}(d_t)/r - \mathrm{E}(p_t)]\}\,\mathrm{d}t \\
&= \{r[p_t - \mathrm{E}(p_t)] - [d_t - \mathrm{E}(d_t)]\}\,\mathrm{d}t.
\end{aligned}$$

Combining the latter with (7) yields

$$p_t - \mathrm{E}(p_t) = \frac{1}{2r}[d_t - \mathrm{E}(d_t)].$$

Taking the stochastic differential obtains

$$
\begin{aligned}
\mathrm{E}_t\left(\mathrm{d}d_t\right) &= 2r\,\mathrm{E}_t\left(\mathrm{d}p_t\right) \\
&= 2r\left\{-r\left[p_t - \mathrm{E}\left(p_t\right)\right]\mathrm{d}t\right\} \\
&= -2r^2\left\{\frac{1}{2r}\left[d_t - \mathrm{E}\left(d_t\right)\right]\right\}\mathrm{d}t \\
&= -r\left[d_t - \mathrm{E}\left(d_t\right)\right]\mathrm{d}t,
\end{aligned}
$$

which proves (8).

Understatement of the Uncertainty

A criticism is that the price and the dividend are not stationary, but tend to rise as time passes. The uncertainty about this rate of increase is an important part of the uncertainty. When Shiller detrends the price and the dividend, implicitly he assumes that the long-run rate of increase is known. A change in the expected long-run rate of increase can cause a great change in the price. Consequently the detrending understates the uncertainty about the price and the dividend.

Stationary Stochastic Process

Suppose that a stationary random variable x_t follows the stochastic differential equation

$$\mathrm{d}x_t = \mu_t \, \mathrm{d}t + \sigma \, \mathrm{d}z_t \, ,$$

in which z_t is Wiener-Brownian motion. Equivalently,

$$x_{t+\mathrm{d}t} - x_t = \mu_t \, \mathrm{d}t + \sigma \, \mathrm{d}z_t \, .$$

Taking the unconditional expected value gives

$$
\begin{aligned}
0 &= \mathrm{E}\left(\mathrm{d}x_t\right) - \mathrm{E}\left(\mu_t\right)\mathrm{d}t - \mathrm{E}\left(\sigma\,\mathrm{d}z_t\right) \\
&= 0 - \mathrm{E}\left(\mu_t\right)\mathrm{d}t - 0,
\end{aligned}
$$

so the mean $\mathrm{E}\left(\mu_t\right) = 0$.

Covariance of Level and Mean Change

Taking the unconditional variance gives

$$
\begin{aligned}
\operatorname{Var}(x_{t+\mathrm{d}t}) &= \operatorname{Var}(x_t + \mu_t\,\mathrm{d}t + \sigma\,\mathrm{d}z_t) \\
&= \operatorname{Var}(x_t) + \operatorname{Var}(\mu_t)(\mathrm{d}t)^2 + \operatorname{Var}(\sigma\,\mathrm{d}z_t) \\
&\quad + 2\operatorname{Cov}(x_t, \mu_t)\,\mathrm{d}t + 2\operatorname{Cov}(x_t, \sigma\,\mathrm{d}z_t) \\
&\quad + 2\operatorname{Cov}(\mu_t\,\mathrm{d}t, \sigma\,\mathrm{d}z_t) \\
&= \operatorname{Var}(x_t) + \sigma^2\,\mathrm{d}t + 2\operatorname{Cov}(x_t, \mu_t)\,\mathrm{d}t.
\end{aligned}
$$

By stationarity, $\operatorname{Var}(x_{t+\mathrm{d}t}) = \operatorname{Var}(x_t)$, which yields the following lemma.

Lemma 2 (Covariance of Level and Mean Change)

$$\mathrm{Cov}\left(x_t, \mu_t\right) = -\frac{1}{2}\sigma^2.$$

The covariance of the level and the mean change must be negative.

References

[1] Robert J. Shiller. Do stock prices move too much to be justified by subsequent changes in dividends? *American Economic Review*, 71:421–436, June 1981. 1, 5