Two-State Option Pricing

Rendleman and Bartter [1] present a simple two-state model of option pricing. The states of the world evolve like the branches of a tree. Given the current state, there are two possible states next period.

Using an arbitrage argument, one prices the option by working backwards from the future to the present.

Notation

Let the subscript 0 denote the current state. Subscripts 1 and 2 denote the two possible states that may occur in the future.

A stock worth s_0 in the current period is worth either s_1 or s_2 in the future period. A call worth c_0 in the current period is worth either c_1 or c_2 in the future period.

There is also a risk-free asset.

Numerical Example

We illustrate the model by a numerical example.

A stock worth $s_0 = 100$ in the current period is worth either $s_1 = 120$ or $s_2 = 90$ in the future period.

For simplicity, suppose that the risk-free rate of return is zero.

Call

Suppose that the call expires in the future period, with exercise price 105. Thus the call is worth

$$c_1 = s_1 - 105 = 15$$

if state 1 occurs (the call is exercised), and

$$c_2 = 0$$

if state 2 occurs (the call is not exercised).

We use an arbitrage argument to find the current price c_0.

Black-Scholes Argument

Black and Scholes argue that one can form a risk-free portfolio from the stock and the call. The difference in stock value between the two future states is

$$s_1 - s_2 = 120 - 90 = 30,$$

and the difference in call value between the two future states is

$$c_1 - c_2 = 15 - 0 = 15.$$

Hedge Ratio

Hence the hedge ratio is

$$\frac{15}{30} = \frac{1}{2}.$$

In the current period, form a risk-free portfolio by buying one share of stock and selling two calls. The net cost of this portfolio is

$$s_0 - 2c_0.$$

Risk-Free Portfolio

If state 1 occurs, the portfolio is worth

$$s_1 - 2c_1 = 120 - 2 \times 15 = 90.$$

If state 2 occurs, the portfolio is worth

$$s_2 - 2c_2 = 90 - 2 \times 0 = 90.$$

The portfolio is indeed risk-free.

Call Price for No Arbitrage

If there is no opportunity for arbitrage profit, the rate of return on this portfolio must equal the risk-free rate of return:

$$(s_0 - 2c_0) \times (1 + \text{risk-free return}) = 90.$$

Hence

$$(100 - 2c_0) \times (1 + 0) = 90,$$

so

$$c_0 = 5.$$

Pricing Kernel

One obtains the same call price via the pricing kernel (stochastic discount factor).

Let p_i denote the state price of one dollar in state i. Since there are two states and two assets (the stock and the risk-free asset), there exists a unique portfolio of these assets having payoff 1 in state i and payoff 0 in the other state. The state price p_i is the cost of this portfolio.

For an asset having payoff $\$_i$ in state i, its price must be

$$p_1\$_1 + p_2\$_2,$$

to avoid an opportunity for profitable arbitrage.

For the stock,

$$s_0 = p_1 s_1 + p_2 s_2,$$

so

$$100 = 120 p_1 + 90 p_2.$$

For the risk-free asset,

$$1 = p_1 + p_2,$$

as the risk-free rate of return is zero.

Solving yields the state prices

$$p_1 = \frac{1}{3}$$
$$p_2 = \frac{2}{3}.$$

Pricing the call via the state prices gives

$$c_0 = p_1 c_1 + p_2 c_2 = \frac{1}{3} \times 15 + \frac{2}{3} \times 0 = 5,$$

identical to the result found above.

Probability and Mean Return

The probability of each state is irrelevant to the call price. This result is natural, since the call pricing is based on an arbitrage argument, and whether there is an arbitrage opportunity is independent of the probability of each state.

The irrelevance of the probability corresponds to the property of the Black-Scholes model that the mean rate of return on the stock is irrelevant. In the two-state model, changing the probability of the two states changes the mean rate of return but has no effect on the call price.

The Black-Scholes Model as a Limit

Wiener-Brownian motion can be derived as the limit of a binomial random walk. Using this relationship, by a complicated argument it is possible to derive the Black-Scholes formula for the call price by taking the limit of the call price derived in the two-state model.

References

[1] Richard J. Rendleman, Jr. and Brit J. Bartter. Two-state option pricing. *Journal of Finance*, XXXIV(5):1093–1110, December 1979. HG1J6.