Asset Return

Definition 1 *If a one-dollar investment in an asset at time t is worth*

$$1+\frac{\mathrm{d}a}{a}$$

at time $t+\mathrm{d}t$, then the stochastic differential

$$\frac{\mathrm{d}a}{a}$$

is the return on the asset.

For example, consider the return

$$\frac{\mathrm{d}a}{a} = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}z.$$

Then the expected return from t to $t+\mathrm{d}t$ is $\mu\,\mathrm{d}t$, and expected *rate* of return (the expected return per unit time) is μ. The variance of the return is $\sigma^2\,\mathrm{d}t$, so the variance is σ^2 per unit time.

Portfolio

If an investor with wealth w_t at time t buys a portfolio with return $\mathrm{d}a/a$, then at time $t+\mathrm{d}t$ his wealth is

$$w_{t+\mathrm{d}t} = w_t \left(1 + \frac{\mathrm{d}a}{a}\right),$$

so the change in wealth is

$$\mathrm{d}w_t = w_{t+\mathrm{d}t} - w_t = w_t \frac{\mathrm{d}a}{a}.$$

Expected Utility

The utility of end-of-period wealth is

$$\begin{aligned} u\left(w_{t+\mathrm{d}t}\right) &= u\left(w_t\right) + \mathrm{d}u\left(w_t\right) \\ &= u\left(w_t\right) + u'\left(w_t\right)\mathrm{d}w_t + \frac{1}{2}u''\left(w_t\right)\left(\mathrm{d}w_t\right)^2 \\ &= u\left(w_t\right) + u'\left(w_t\right)\left(w_t\frac{\mathrm{d}a}{a}\right) + \frac{1}{2}u''\left(w_t\right)\left(w_t\frac{\mathrm{d}a}{a}\right)^2. \end{aligned}$$

The expected utility is

$$
\begin{aligned}
&\mathrm{E}\left[u\left(w_{t+\mathrm{d}t}\right)\right] \\
&= \mathrm{E}\left[u\left(w_t\right)+w_t u'\left(w_t\right)\left(\frac{\mathrm{d}a}{a}\right)+\frac{1}{2}w_t^2 u''\left(w_t\right)\left(\frac{\mathrm{d}a}{a}\right)^2\right] \\
&= u\left(w_t\right)+w_t u'\left(w_t\right)\mathrm{E}\left(\frac{\mathrm{d}a}{a}\right)+\frac{1}{2}w_t^2 u''\left(w_t\right)\mathrm{Var}\left(\frac{\mathrm{d}a}{a}\right) \\
&= u\left(w_t\right)+w_t u'\left(w_t\right)\left[\mathrm{E}\left(\frac{\mathrm{d}a}{a}\right)+\frac{1}{2}\frac{w_t u''\left(w_t\right)}{u'\left(w_t\right)}\mathrm{Var}\left(\frac{\mathrm{d}a}{a}\right)\right].
\end{aligned}
$$

Make a linear transformation of utility so that $u(w_t) = 0$ and $w_t u'(w_t) = 1$.

Then the expected utility is the expression in brackets,

$$\mathrm{E}\left(\frac{\mathrm{d}a}{a}\right) - \frac{1}{2}\alpha \mathrm{Var}\left(\frac{\mathrm{d}a}{a}\right), \tag{1}$$

in which α is the relative risk aversion.

In this derivation,

$$\left(\frac{\mathrm{d}a}{a}\right)^2 = (\mu\,\mathrm{d}t + \sigma\,\mathrm{d}z)^2 = (\sigma\,\mathrm{d}z)^2 = \sigma^2\,\mathrm{d}t = \mathrm{Var}\left(\frac{\mathrm{d}a}{a}\right).$$

In contrast, in discrete time no such simple result exists. In the square

$$(\mu\,\Delta t + \sigma\,\Delta z)^2,$$

the term $\mu\,\Delta t$ does not disappear, and $(\Delta z)^2$ is stochastic. Furthermore, μ and σ may change during the interval Δt.

Two-Asset Portfolio Choice

Consider portfolio choice with a risk-free asset and a single risky asset. The return on the risk-free asset is

$$r\,\mathrm{d}t.$$

The return on the risky asset is

$$(r+m)\,\mathrm{d}t+s\,\mathrm{d}z.$$

Here $m>0$ is the expected excess return, and $s>0$ is the standard deviation.

Portfolio Return

If one invests the fraction f of wealth in the risky asset and the fraction $1-f$ in the risk-free asset, then the return on the portfolio is the returns weighted by the fractions,

$$(1-f)(r\,\mathrm{d}t)+f\left[(r+m)\,\mathrm{d}t+s\,\mathrm{d}z\right]=(r+fm)\,\mathrm{d}t+fs\,\mathrm{d}z.$$

The mean rate of return is $\mu=r+fm$ and the variance of the return is $f^2s^2\,\mathrm{d}t$. We refer to $\sigma=fs$ as the standard deviation.

Efficient Frontier

As f varies, one obtains the various possible combinations of standard deviation and mean. In (standard deviation, mean)-space, one obtains a straight line, with intercept r and slope m/s.

We refer to this straight line as the *efficient frontier*.

The value $f > 1$ signifies leveraged investment: one borrows at the risk-free interest rate and then invests this borrowing plus the initial wealth in the risky asset.

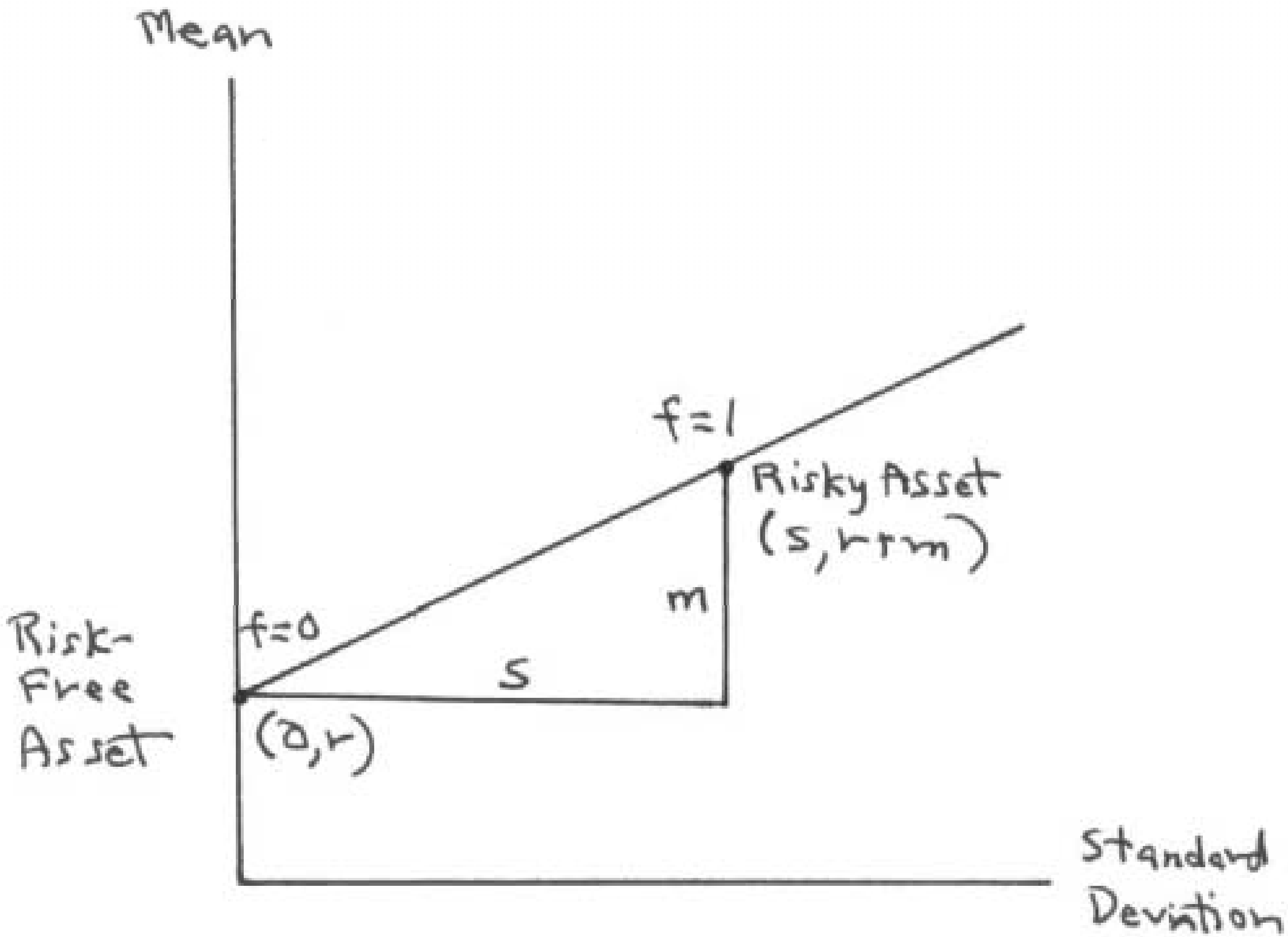

Figure 1: Possibilities

Expected Utility Maximization

The efficient frontier shows the possibilities and is akin to a budget line.

By (1), the optimum portfolio choice maximizes

$$\mu - \frac{1}{2}\alpha\sigma^2.$$

In (standard deviation, mean)-space, the indifference curves are parabolas.

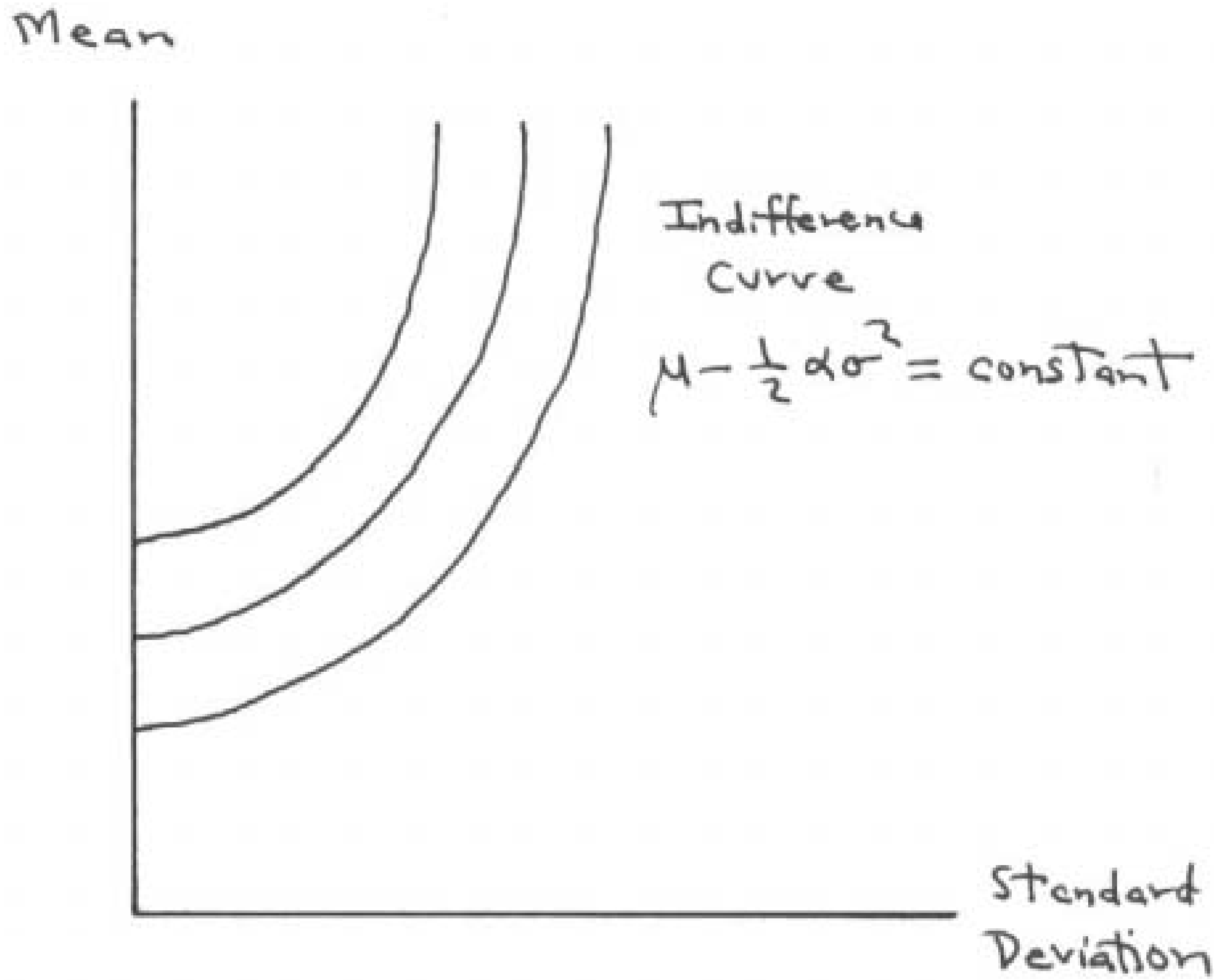

Figure 2: Indifference Curves

Slope of Indifference Curve

Along an indifference curve

$$\mu - \frac{1}{2}\alpha\sigma^2 = \text{constant}.$$

Viewing μ as a function of σ, differentiate with respect to σ,

$$\frac{\mathrm{d}\mu}{\mathrm{d}\sigma} - \alpha\sigma = 0.$$

Hence the slope of an indifference curve is

$$\frac{\mathrm{d}\mu}{\mathrm{d}\sigma} = \alpha\sigma.$$

Indifference Curve Tangent to Efficient Frontier

The optimum portfolio choice is to pick f so an indifference curve is tangent to the efficient frontier.

Setting the slope of the efficient frontier equal to the slope of the indifference curve gives

$$\frac{m}{s} = \alpha\sigma = \alpha f s.$$

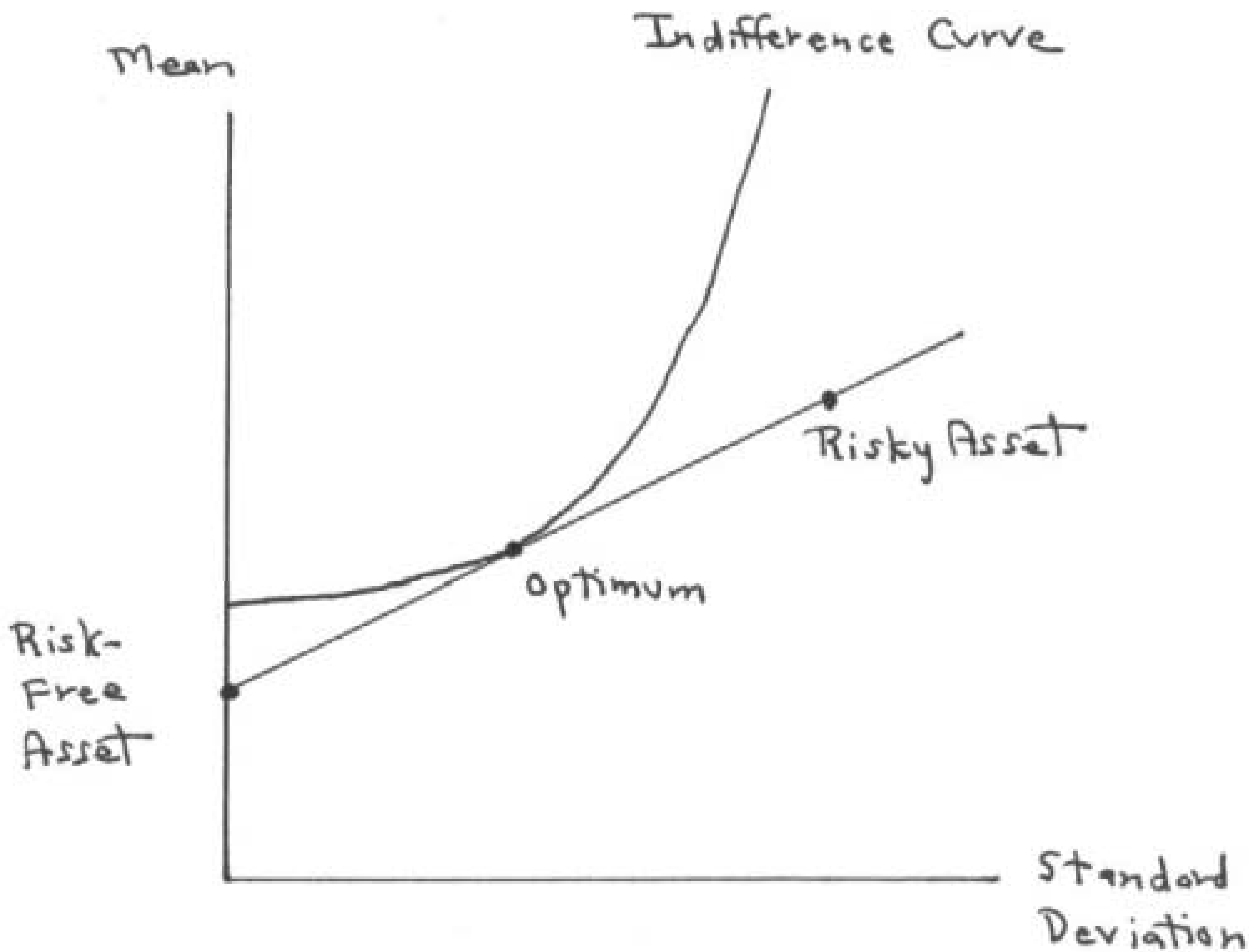

Figure 3: Optimum Portfolio Choice

Optimum Portfolio Choice

Solving for f gives

$$f = \frac{m}{\alpha s^2}.$$

This formula makes sense intuitively. As m increases, as α decreases, or as s decreases, the investment in the risky asset rises.