**Term Structure**

Consider a simplification of the model of Vasicek [1] of the term structure of interest rates.

The short-term, risk-free interest rate $r$ follows a random walk,

$$dr = \rho \, dz.$$ 

Let $P(\tau, r)$ denote the price of a risk-free pure discount bond worth one dollar at its maturity in $\tau$ years. Of course $P(0, r) = 1$. We wish to solve for the equilibrium price.

**Yield to Maturity**

Let $R(\tau, r)$ denote the yield to maturity on the $\tau$-year bond. By definition,

$$P(\tau, r) = e^{-\tau R(\tau, r)},$$

so

$$R(\tau, r) = -\frac{1}{\tau} \ln P(\tau, r).$$

**Expectations Theory of the Term Structure**

The standard model of the term structure is the expectations theory, which argues that the long-term interest rate is the average of the current and expected future short-term interest rates.

Here the expected future short-term rate is just the current short-term rate, so

$$R(\tau, r) = r$$

according to the expectations theory. Hence

$$P(\tau, r) = e^{-r \tau}.$$ 

**Return**

The price of a bond at time $t$ maturing at time $T$ is $P(T - t, r)$. The return on the bond is the price change $dP/P$.

By Itô’s formula,

$$dP = -\frac{\partial P}{\partial \tau} dt + \frac{\partial P}{\partial r} dr + \frac{1}{2} \rho^2 \frac{\partial^2 P}{\partial r^2} (dr)^2$$

($\tau$ falls as $t$ rises)

$$= -\frac{\partial P}{\partial \tau} dt + \frac{\partial P}{\partial r} \rho dz + \frac{1}{2} \rho^2 \frac{\partial^2 P}{\partial r^2} (\rho dz)^2$$

$$= \left( -\frac{\partial P}{\partial \tau} + \frac{1}{2} \rho^2 \frac{\partial^2 P}{\partial r^2} \right) dt + \rho \frac{\partial P}{\partial r} dz.$$ 

**Market Equilibrium**

For market equilibrium, assume that all bonds must have expected rate of return $r$:

$$r \, dt = E_t \left( \frac{dP}{P} \right) = \frac{1}{P} \left( -\frac{\partial P}{\partial \tau} + \frac{1}{2} \rho^2 \frac{\partial^2 P}{\partial r^2} \right) dt.$$ 

**Term-Structure Equation**

We wish to solve the term-structure equation

$$rP = -\frac{\partial P}{\partial \tau} + \frac{1}{2} \rho^2 \frac{\partial^2 P}{\partial r^2},$$

subject to the boundary condition $P(0, r) = 1$. 

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### Constant Interest Rate

The special case $\rho = 0$ implies a constant interest rate. The term-structure equation is then

$$rP = -\frac{\partial P}{\partial \tau},$$

with solution

$$P(\tau, r) = e^{-rt}.$$

The yield to maturity is

$$R(\tau, r) = r,$$

in agreement with the expectations theory.

### General Solution

The general solution is

$$P(\tau, r) = e^{-rt + \frac{1}{2} \rho^2 \tau^2},$$

which one verifies by substituting into the term-structure equation.

### Return

The return is

$$\frac{dP}{P} = \left( -\frac{1}{P} \frac{\partial P}{\partial \tau} + \frac{1}{2} \rho^2 \frac{1}{P} \frac{\partial^2 P}{\partial r^2} \right) dt + \rho \frac{1}{P} \frac{\partial P}{\partial r} dz$$

$$= \left[ \left( r - \frac{1}{2} \rho^2 \tau^2 \right) + \frac{1}{2} \rho^2 \tau^2 \right] dt - \rho \tau dz$$

$$= r dt - \tau dr.$$

An increase in $r$ reduces $P$, and the standard deviation of the return is proportional to the term to maturity.

### Yield to Maturity

The yield to maturity is

$$R(\tau, r) = -\frac{1}{\tau} \ln P(\tau, r) = r - \frac{1}{6} \rho^2 \tau^2.$$

### Risk Premium

Alternatively, one might allow the possibility of a risk premium. The stochastic differential for the price takes the form

$$\frac{dP}{P} = m(\tau, r) dt + s(\tau, r) dz.$$

The returns for the different bonds are perfectly correlated, since each involves the same instantaneous error $dz$. 

No Arbitrage

Consequently there will be an arbitrage opportunity unless the risk premium is proportional to the standard deviation:

\[ m(\tau, r) - r \propto s(\tau, r). \]

Let \( q \) denote the proportionality factor.

Constant Risk Premium

For constant \( q \), the bond price is

\[ P(\tau, r) = e^{-r \tau - \frac{1}{2} q \rho \tau^2 + \frac{1}{3} \rho^2 \tau^3}. \]

The yield to maturity is

\[ R(\tau, r) = -\frac{1}{\tau} \ln P(\tau, r) = r + q \rho \tau - \frac{1}{6} \rho^2 \tau^2. \]

Return

The return is

\[
\frac{dP}{P} = \left( -\frac{1}{P} \frac{\partial P}{\partial \tau} + \frac{1}{2} \rho^2 \frac{1}{P} \frac{\partial^2 P}{\partial r^2} \right) dt + \rho \frac{1}{P} \frac{\partial P}{\partial r} dz \\
= \left[ \left( r + q \rho \tau - \frac{1}{2} \rho^2 \tau^2 \right) + \frac{1}{2} \rho^2 \tau^2 \right] dt - \rho \tau dz \\
= (r + q \rho \tau) dt - \tau dr.
\]

Term-Structure Equation

The term structure equation is

\[ m(\tau, r) - r = q s(\tau, r), \]

which takes the form

\[ \left( -\frac{1}{P} \frac{\partial P}{\partial \tau} + \frac{1}{2} \rho^2 \frac{1}{P} \frac{\partial^2 P}{\partial r^2} \right) - r = q \left( -\rho \frac{1}{P} \frac{\partial P}{\partial r} \right). \]

Hence the term-structure equation (1) changes to

\[ rP = -\frac{\partial P}{\partial \tau} + q \rho \frac{\partial P}{\partial r} + \frac{1}{2} \rho^2 \frac{\partial^2 P}{\partial r^2}. \]

To express \( q \) as a function of \( r \) would be a natural model.

References