Term Structure

Consider a simplification of the model of Vasicek [1] of the term structure of interest rates.

The short-term, risk-free interest rate r follows a random walk,

$$\mathrm{d}r = \rho\,\mathrm{d}z.$$

Let $P(\tau, r)$ denote the price of a risk-free pure discount bond worth one dollar at its maturity in τ years. Of course $P(0, r) = 1$. We wish to solve for the equilibrium price.

Yield to Maturity

Let $R(\tau, r)$ denote the yield to maturity on the τ-year bond. By definition,

$$P(\tau, r) = \mathrm{e}^{-\tau R(\tau, r)},$$

so

$$R(\tau, r) = -\frac{1}{\tau} \ln P(\tau, r).$$

Expectations Theory of the Term Structure

The standard model of the term structure is the expectations theory, which argues that the long-term interest rate is the average of the current and expected future short-term interest rates.

Here the expected future short-term rate is just the current short-term rate, so

$$R(\tau, r) = r$$

according to the expectations theory. Hence

$$P(\tau, r) = \mathrm{e}^{-r\tau}.$$

Return

The price of a bond at time t maturing at time T is $P(T-t,r)$. The return on the bond is the price change $\mathrm{d}P/P$.

By Itô's formula,

$$\begin{aligned}
\mathrm{d}P &= -\frac{\partial P}{\partial \tau}\mathrm{d}t + \frac{\partial P}{\partial r}\mathrm{d}r + \frac{1}{2}\frac{\partial^2 P}{\partial r^2}(\mathrm{d}r)^2 \quad (\tau \text{ falls as } t \text{ rises}) \\
&= -\frac{\partial P}{\partial \tau}\mathrm{d}t + \frac{\partial P}{\partial r}\rho\,\mathrm{d}z + \frac{1}{2}\frac{\partial^2 P}{\partial r^2}(\rho\,\mathrm{d}z)^2 \\
&= \left(-\frac{\partial P}{\partial \tau} + \frac{1}{2}\rho^2\frac{\partial^2 P}{\partial r^2}\right)\mathrm{d}t + \rho\frac{\partial P}{\partial r}\mathrm{d}z.
\end{aligned}$$

Market Equilibrium

For market equilibrium, assume that all bonds must have expected rate of return r:

$$r\,\mathrm{d}t = \mathrm{E}_t\left(\frac{\mathrm{d}P}{P}\right) = \frac{1}{P}\left(-\frac{\partial P}{\partial \tau} + \frac{1}{2}\rho^2\frac{\partial^2 P}{\partial r^2}\right)\mathrm{d}t.$$

Term-Structure Equation

We wish to solve the term-structure equation

$$rP = -\frac{\partial P}{\partial \tau} + \frac{1}{2}\rho^2 \frac{\partial^2 P}{\partial r^2}, \tag{1}$$

subject to the boundary condition $P(0, r) = 1$.

Constant Interest Rate

The special case $\rho = 0$ implies a constant interest rate. The term-structure equation is then

$$rP = -\frac{\partial P}{\partial \tau},$$

with solution

$$P(\tau, r) = \mathrm{e}^{-r\tau}.$$

The yield to maturity is

$$R(\tau, r) = r,$$

in agreement with the expectations theory.

General Solution

The general solution is

$$P(\tau, r) = \mathrm{e}^{-r\tau + \frac{1}{6}\rho^2\tau^3},$$

which one verifies by substituting into the term-structure equation.

Return

The return is

$$\frac{\mathrm{d}P}{P} = \left(-\frac{1}{P}\frac{\partial P}{\partial \tau} + \frac{1}{2}\rho^2\frac{1}{P}\frac{\partial^2 P}{\partial r^2}\right)\mathrm{d}t + \rho\frac{1}{P}\frac{\partial P}{\partial r}\,\mathrm{d}z$$

$$= \left[\left(r - \frac{1}{2}\rho^2\tau^2\right) + \frac{1}{2}\rho^2\tau^2\right]\mathrm{d}t - \rho\,\tau\,\mathrm{d}z$$

$$= r\,\mathrm{d}t - \tau\,\mathrm{d}r.$$

An increase in r reduces P, and the standard deviation of the return is proportional to the term to maturity.

Yield to Maturity

The yield to maturity is

$$R(\tau, r) = -\frac{1}{\tau} \ln P(\tau, r) = r - \frac{1}{6} \rho^2 \tau^2.$$

The yield to maturity at time t on a bond maturing at time T is $R(T-t,r)$, which follows the stochastic differential equation

$$\begin{aligned} \mathrm{d}R &= -\frac{\partial R}{\partial \tau}\mathrm{d}t + \frac{\partial R}{\partial r}\mathrm{d}r + \frac{1}{2}\frac{\partial^2 R}{\partial r^2}(\mathrm{d}r)^2 \\ &= -\left[-\frac{1}{3}\rho^2(T-t)\right]\mathrm{d}t + 1\,\mathrm{d}r + \frac{1}{2}0\,(\mathrm{d}r)^2 \\ &= \frac{1}{3}\rho^2(T-t)\,\mathrm{d}t + \mathrm{d}r. \end{aligned}$$

Risk Premium

Alternatively, one might allow the possibility of a risk premium. The stochastic differential for the price takes the form

$$\frac{\mathrm{d}P}{P} = m(\tau, r)\,\mathrm{d}t + s(\tau, r)\,\mathrm{d}z.$$

The returns for the different bonds are perfectly correlated, since each involves the same instantaneous error dz.

No Arbitrage

Consequently there will be an arbitrage opportunity unless the risk premium is proportional to the standard deviation:

$$m(\tau, r) - r \propto s(\tau, r).$$

Let q denote the proportionality factor.

Term-Structure Equation

The term structure equation is

$$m(\tau, r) - r = q s(\tau, r),$$

which takes the form

$$\left(-\frac{1}{P}\frac{\partial P}{\partial \tau} + \frac{1}{2}\rho^2 \frac{1}{P}\frac{\partial^2 P}{\partial r^2} \right) - r = q \left(-\rho \frac{1}{P}\frac{\partial P}{\partial r} \right).$$

Hence the term-structure equation (1) changes to

$$rP = -\frac{\partial P}{\partial \tau} + q\rho \frac{\partial P}{\partial r} + \frac{1}{2}\rho^2 \frac{\partial^2 P}{\partial r^2}.$$

To express q as a function of r would be a natural model.

Constant Risk Premium

For constant q, the bond price is

$$P(\tau, r) = \mathrm{e}^{-r\tau - \frac{1}{2}q\rho\tau^2 + \frac{1}{6}\rho^2\tau^3}.$$

The yield to maturity is

$$R(\tau, r) = -\frac{1}{\tau}\ln P(\tau, r) = r + q\rho\tau - \frac{1}{6}\rho^2\tau^2.$$

Return

The return is

$$\begin{aligned}
\frac{\mathrm{d}P}{P} &= \left(-\frac{1}{P}\frac{\partial P}{\partial \tau} + \frac{1}{2}\rho^2 \frac{1}{P}\frac{\partial^2 P}{\partial r^2}\right)\mathrm{d}t + \rho\frac{1}{P}\frac{\partial P}{\partial r}\,\mathrm{d}z \\
&= \left[\left(r + q\rho\tau - \frac{1}{2}\rho^2\tau^2\right) + \frac{1}{2}\rho^2\tau^2\right]\mathrm{d}t - \rho\tau\,\mathrm{d}z \\
&= (r + q\rho\tau)\,\mathrm{d}t - \tau\,\mathrm{d}r.
\end{aligned}$$

References

[1] Oldrich Vasicek. An equilibrium characterization of the term structure. *Journal of Financial Economics*, 5(2):177–188, November 1977. HB1J69X.