Portfolio Choice

Let us model portfolio choice formally in Euclidean space.

There are n assets, and the portfolio space $\mathbf{X} = \mathbf{R}^n$. A vector $\boldsymbol{x} \in \mathbf{X}$ is a portfolio. Even though we like to see a vector as coordinate-free, there is a natural orthonormal basis with elements $\boldsymbol{x}_j$: buy one unit of asset j and zero units of all other assets. Since $\langle \boldsymbol{x}_i, \boldsymbol{x}_j \rangle := \delta_{ij}$, the basis is orthonormal. Expressed in terms of the natural basis,

$$\boldsymbol{x} = \sum_j x_j \boldsymbol{x}_j,$$

in which x_j is the quantity of asset j in the portfolio.

Portfolio Cost

The cost of a portfolio is a linear function $\langle \boldsymbol{v}, \boldsymbol{x} \rangle$, and we refer to $\boldsymbol{v}$ as the asset-price vector. Expressed in terms of the natural basis,

$$\boldsymbol{v} = \sum_j v_j \boldsymbol{x}_j,$$

in which v_j is the price of asset j.

State Space

The m-dimensional state space $\mathbf{Y}$ describes the m possible states of the world. A vector $\boldsymbol{y} \in \mathbf{Y}$ specifies the dollar payoff in each state. Even though we like to see a vector as coordinate-free, there is a natural basis with elements $\boldsymbol{y}_i$: the dollar payoff is one dollar in state i and zero dollars otherwise.

Expressed in terms of the natural basis,

$$\boldsymbol{y} = \sum_i y_i \boldsymbol{y}_i,$$

in which y_i is the payoff in state i.

Inner Product

Definition 1 (State-Space Inner Product) *The inner product of two payoff vectors is their second noncentral moment.*

In terms of the natural basis, the inner product is

$$\langle \boldsymbol{y}_i, \boldsymbol{y}_j \rangle := \pi_i \delta_{ij},$$

in which $\pi_i > 0$ is the probability that state i occurs. Note that the natural basis for the state space is *not* orthonormal.

Payoff

Definition 2 (Payoff Transformation) *The payoff transformation* A *is a linear transformation* $A : \mathbf{X} \to \mathbf{Y}$ *such that* $\boldsymbol{x} \mapsto \boldsymbol{y} = A\boldsymbol{x}$ *is the payoff on portfolio* $\boldsymbol{x}$.

Matrix Representation

The matrix representation A_{ij} in terms of the natural bases is the payoff of asset j in state i,

$$A\boldsymbol{x}_j = \sum_i A_{ij}\boldsymbol{y}_i. \tag{1}$$

Expressing $\boldsymbol{y} = \mathsf{A}\boldsymbol{x}$ in coordinates says

$$\begin{aligned}\sum_i y_i \boldsymbol{y}_i &= \mathsf{A}\sum_j x_j \boldsymbol{x}_j \\ &= \sum_j x_j \left(\mathsf{A}\boldsymbol{x}_j\right) \\ &= \sum_j x_j \sum_i A_{ij} \boldsymbol{y}_i, \text{ by (1)} \\ &= \sum_i \left(\sum_j A_{ij} x_j\right) \boldsymbol{y}_i.\end{aligned}$$

Thus the coordinates transform in the standard matrix form

$$\begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{m1} & \cdots & A_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

Adjoint

Lemma 3 (Adjoint of the Payoff Transformation) *The adjoint*

$$A^\top \boldsymbol{y}_i = \sum_j \pi_i A_{ij} \boldsymbol{x}_j.$$

Since the basis in the state space is not orthonormal, the matrix representation of the adjoint is not the transpose of the matrix representation of the payoff transformation. Instead, the state probability is an extra factor.

Invoking the adjoint equality proves the lemma:

$$\begin{aligned}\left\langle \boldsymbol{A}^{\top}\boldsymbol{y}_k, \boldsymbol{x}\right\rangle &= \langle \boldsymbol{y}_k, \boldsymbol{A}\boldsymbol{x}\rangle \\ &= \left\langle \boldsymbol{y}_k, \sum_{i.j} x_j A_{ij} \boldsymbol{y}_i \right\rangle \\ &= \sum_j x_j A_{kj} \pi_k \\ &= \left\langle \sum_j \pi_k A_{kj} \boldsymbol{x}_j, \sum_j x_j \boldsymbol{x}_j \right\rangle \\ &= \left\langle \sum_j \pi_k A_{kj} \boldsymbol{x}_j, \boldsymbol{x} \right\rangle.\end{aligned}$$

Stochastic Discount Factor

Definition 4 *A stochastic discount factor is a state vector such that for any portfolio its cost is the expected value of the stochastic discount factor times its payoff.*

Thus $\boldsymbol{y}$ is a stochastic discount factor if and only if

$$\langle \boldsymbol{v}, \boldsymbol{x} \rangle = \langle \boldsymbol{y}, \boldsymbol{A}\boldsymbol{x} \rangle ,$$

for any $\boldsymbol{x}$. On the left-hand side, the inner product is in the portfolio space; on the right-hand side, the inner product is in the state space.

Equivalently,

$$\begin{aligned} 0 &= \langle \boldsymbol{v}, \boldsymbol{x} \rangle - \langle \boldsymbol{y}, A\boldsymbol{x} \rangle \\ &= \langle \boldsymbol{v}, \boldsymbol{x} \rangle - \left\langle A^\top \boldsymbol{y}, \boldsymbol{x} \right\rangle \\ &= \left\langle \boldsymbol{v} - A^\top \boldsymbol{y}, \boldsymbol{x} \right\rangle. \end{aligned}$$

That this condition must hold for any $\boldsymbol{x}$ yields the following.

Theorem 5 (Stochastic Discount Factor) *A stochastic discount factor is a state vector* $\boldsymbol{y}$ *such that*

$$\boldsymbol{v} = A^{\top}\boldsymbol{y}. \tag{2}$$

Matrix Representation

We verify using coordinates that equation (2) expresses an asset price as the expected value of the stochastic discount factor times the payoff.

Expressing (2) in coordinates,

$$
\begin{aligned}
\sum_j v_j \boldsymbol{x}_j &= A^\top \sum_i y_i \boldsymbol{y}_i \\
&= \sum_i y_i \left(A^\top \boldsymbol{y}_i \right) \\
&= \sum_i y_i \left(\sum_j \pi_i A_{ij} \boldsymbol{x}_j \right) \\
&= \sum_j \left(\sum_i \pi_i A_{ij} y_i \right) \boldsymbol{x}_j ,
\end{aligned}
$$

Hence

$$v_j = \sum_i \pi_i A_{ij} y_i.$$

The asset price v_j is the sum of terms of the form probability π_i times the payoff A_{ij} times the stochastic discount factor y_i.

In standard matrix form,

$$\begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} \pi_1 A_{11} & \cdots & \pi_m A_{m1} \\ \vdots & \ddots & \vdots \\ \pi_1 A_{1n} & \cdots & \pi_m A_{mn} \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_m \end{bmatrix}.$$

Law of One Price

Definition 6 (Law of One Price) *Two assets having the same payoff in each state must sell for the same price.*

The law of one price is a necessary condition for market equilibrium.

Fundamental Theorem of Linear Algebra

The law of one price says

$$A\boldsymbol{x} = \mathbf{0} \Rightarrow \langle \boldsymbol{v}, \boldsymbol{x} \rangle = 0.$$

The geometric/linear algebra interpretation is that $\boldsymbol{v}$ is orthogonal to the null space of A. The fundamental theorem of linear algebra says that the null space of A is the orthogonal complement of the range of $A^\top$. Hence $\boldsymbol{v} = A^\top \boldsymbol{y}$ for some $\boldsymbol{y}$—there is a stochastic discount factor. Working backwards shows the converse, and we have the following theorem.

Theorem 7 (Law of One Price) *The law of one price holds if and only if there exists a stochastic discount factor.*

That a stochastic discount factor implies the law of one price is obvious, so what is interesting is the converse.

Risk-Free Asset

Define $\mathbf{1}$ as a state vector having one dollar payoff in each state. By definition, $\langle \mathbf{1}, \mathbf{1} \rangle = 1$, as the state probabilities sum to one.

That there exists a risk-free asset means that $\mathbf{1}$ lies in the payoff space $\mathrm{R}(\mathsf{A})$. A risk-free portfolio having payoff $\mathbf{1}$ costs $\langle \boldsymbol{y}, \mathbf{1} \rangle$, which must be positive to avoid an arbitrage opportunity.

Risk Premium

Let $\boldsymbol{\xi}$ denote the excess return on an asset. Necessarily $\langle \boldsymbol{y}, \boldsymbol{\xi} \rangle = 0$ for any excess return.

By definition, covariance is the second noncentral moment less the product of the expected values,

$$\mathrm{Cov}\left(\boldsymbol{y}, \boldsymbol{\xi}\right) := \langle \boldsymbol{y}, \boldsymbol{\xi} \rangle - \langle \mathbf{1}, \boldsymbol{y} \rangle \langle \mathbf{1}, \boldsymbol{\xi} \rangle ,$$

and the following theorem is immediate.

Theorem 8 (Risk Premium) *For any asset, its risk premium*

$$\mathrm{E}(\boldsymbol{\xi}) = -\frac{\mathrm{Cov}(\boldsymbol{y}, \boldsymbol{\xi})}{\mathrm{E}(\boldsymbol{y})},$$

for any stochastic discount factor $\boldsymbol{y}$.

If the payoff on an asset tends to be high where its value is low (where the stochastic discount factor is low), then its expected rate-of-return must be high.