Stochastic Difference Equation

Let z_t denote a discrete-time, normal random walk.

Definition 1 *The stochastic difference equation*

$$\Delta x_t = m\,\Delta t + s\,\Delta z_t$$

means that the change Δx_t in x_t follows

$$\Delta x_t \sim \mathrm{N}(m\,\Delta t, s^2\,\Delta t).$$

Here $m\,\Delta t$ is the mean change, $s\,\Delta z_t$ is the error, and $s^2\,\Delta t$ is the variance of the error.

Solution to the Stochastic Difference Equation

Definition 2 (Solution to an Initial Value Problem) *Given an initial value* x_0 *together with values the error* $\Delta z_0, \Delta z_{\Delta t}, \Delta z_{2\Delta t}, \ldots$ *(equivalently, given* $z_{\Delta t}, z_{2\Delta t}, \ldots$*), find* $\Delta x_0, \Delta x_{\Delta t}, \Delta x_{2\Delta t}, \ldots$*(equivalently, find* $x_{\Delta t}, x_{2\Delta t}, \ldots$*).*

Solution

For the stochastic difference equation $\Delta x = m\,\Delta t + s\,\Delta z$, the solution $x_{\Delta t}$ is

$$\begin{aligned} x_{\Delta t} &= x_0 + \Delta x_0 \\ &= x_0 + (m\,\Delta t + s\,\Delta z_0)\,. \end{aligned}$$

Continuing,

$$\begin{aligned} x_{2\Delta t} &= x_{\Delta t} + \Delta x_{\Delta t} \\ &= [x_0 + (m\,\Delta t + s\,\Delta z_0)] + (m\,\Delta t + s\,\Delta z_{\Delta t}) \\ &= x_0 + 2m\,\Delta t + s\,(\Delta z_0 + \Delta z_{\Delta t}) \\ &= x_0 + 2m\,\Delta t + s z_{2\Delta t}\,. \end{aligned}$$

For $t = n\Delta t$,

$$x_t = x_{n\Delta t} = x_0 + nm\,\Delta t + sz_{n\Delta t}.$$

Hence

$$x_t = x_0 + mt + sz_t,$$

so

$$x_t \sim \mathrm{N}(mt, s^2 t).$$

Stochastic Differential Equation

Definition 3 *The stochastic differential equation*

$$\mathrm{d}x_t = m\,\mathrm{d}t + s\,\mathrm{d}z_t$$

is the limit of the stochastic difference equation as $\Delta t \to 0$.

Seeing $\mathrm{d}t$ as an infinitesimal change in time, then

$$\mathrm{d}x_t \sim \mathrm{N}(m\,\mathrm{d}t, s^2\,\mathrm{d}t).$$

Solution to the Stochastic Differential Equation

Definition 4 (Solution to an Initial Value Problem) *Given an initial value x_0 together with values z_t, $t \geq 0$, find x_t, $t \geq 0$.*

Calculation of the Solution

Solve the stochastic difference equation, and take the limit of the solution. The concept is that the limit of the solution is the solution of the limit.

The solution to the stochastic difference equation is

$$x_t = x_0 + mt + sz_t \,.$$

As this solution is independent of Δt, it is also the solution of the stochastic differential equation.

First-Order Autoregression

Consider a first-order autoregression:

$$\Delta x_t = -a x_t \, \Delta t + s \, \Delta z_t \, .$$

Here the mean change $-a x_t \, \Delta t$ is proportional to the length of the time period.

Equivalently,

$$x_{t+\Delta t} = x_t - a x_t \, \Delta t + s \, \Delta z_t = (1 - a\Delta t) \, x_t + s \, \Delta z_t \, .$$

The stochastic process is stationary if and only if $|1 - a\Delta t| < 1$.

Solution to the First-Order Autoregression

For $t = n\Delta t$, the solution is

$$x_t = (1 - a\Delta t)^n x_0 + s\left[(1 - a\Delta t)^{n-1}\Delta z_0 + (1 - a\Delta t)^{n-2}\Delta z_{\Delta t} + \cdots + \Delta z_{t-\Delta t}\right].$$

Stochastic Differential Equation

As $\Delta t \to 0$, the limit of the stochastic difference equation is the stochastic differential equation

$$\mathrm{d}x_t = -ax_t \,\mathrm{d}t + s\,\mathrm{d}z_t \,.$$

In the limit the stochastic difference equation is stationary if and only if $a > 0$, so the stochastic differential equation is stationary if and only if this condition holds.

Using

$$\lim_{\Delta t \to 0} (1 + \Delta t)^{1/\Delta t} = \mathrm{e},$$

one can show that the limit of the solution to the stochastic difference equation is

$$x_t = \mathrm{e}^{-at} x_0 + s \int_0^t \mathrm{e}^{-a(t-\tau)} \mathrm{d}z_\tau .$$

Generalization

The stochastic differential equation

$$\mathrm{d}x_t = m(x_t, t)\,\mathrm{d}t + s(x_t, t)\,\mathrm{d}z_t$$

is the limit of the stochastic difference equation

$$\Delta x_t = m(x_t, t)\,\Delta t + s(x_t, t)\,\Delta z_t\,.$$

One finds the solution to the stochastic differential equation by taking the limit of the solutions to the stochastic difference equation as $\Delta t \to 0$.

An Unexpected Finding

Consider the stochastic differential equation

$$\mathrm{d}x_t = 2z_t \, \mathrm{d}z_t \, ,$$

such that $x_0 = 0$. In non-stochastic calculus, the solution would be $x_t = z_t^2$. However for the stochastic case it turns out that the solution is different.

Solution

We calculate the solution to the stochastic differential equation as the limit of the solution to the corresponding stochastic difference equation,

$$\Delta x_t = 2 z_t \, \Delta z_t \, .$$

We solve the stochastic difference equation iteratively.

For simplicity of notation, define $e_i = \Delta z_{(i-1)\Delta t}$, so

$$z_{n\Delta t} = e_1 + e_2 + \cdots + e_n.$$

Of course e_i is white noise, $e_i \sim \mathrm{N}(0, \Delta t)$.

We have

$$\Delta x_0 = 2z_0 \, \Delta z_0 = 2 \times 0 e_1 = 0,$$

so

$$x_{\Delta t} = 0.$$

Then

$$\Delta x_{\Delta t} = 2z_{\Delta t}\,\Delta z_{\Delta t} = 2e_1e_2,$$

so

$$x_{2\Delta t} = x_{\Delta t} + \Delta x_{\Delta t} = 2e_1e_2.$$

Then

$$\Delta x_{2\Delta t} = 2z_{2\Delta t}\,\Delta z_{2\Delta t} = 2\,(e_1 + e_2)\,e_3,$$

so

$$x_{3\Delta t} = x_{2\Delta t} + \Delta x_{2\Delta t} = 2\,(e_1e_2 + e_1e_3 + e_2e_3)\,.$$

In general,

$$\begin{aligned} x_{n\Delta t} = 2\ \big(e_1 e_2 \quad &+ e_1 e_3 \quad + \cdots \quad + e_1 e_n \\ &+ e_2 e_3 \quad + \cdots \quad + e_2 e_n \\ &\qquad\qquad + \cdots \\ &\qquad\qquad\qquad + e_{n-1} e_n\big), \end{aligned}$$

so

$$x_{n\Delta t} = z_{n\Delta t}^2 - \left(e_1^2 + e_2^2 + \cdots + e_n^2\right).$$

We rewrite the sum of the squared errors as

$$e_1^2 + e_2^2 + \cdots + e_n^2 = t \left\{ \frac{1}{n} \left[\left(\frac{e_1^2}{\Delta t} \right) + \left(\frac{e_2^2}{\Delta t} \right) + \cdots + \left(\frac{e_n^2}{\Delta t} \right) \right] \right\}.$$

Holding $t = n\Delta t$ fixed, take the limit as $\Delta t \to 0$, $n \to \infty$.

The expression in braces is the sample mean of n independent $\chi^2(1)$ variables. By the law of large numbers, the sample mean converges to the true mean 1 as the sample size increases. Hence

$$\lim_{n\to\infty}\left(e_1^2+e_2^2+\cdots+e_n^2\right)=t,$$

so

$$x_t=z_t^2-t$$

is the solution to the stochastic differential equation. Here t is an extra term!