Expected Utility Maximization

Consider an individual with utility function $u(w)$, increasing and concave in wealth w. The individual maximizes expected utility $\mathrm{E}[u(w)]$.

Mean/Variance Choice and Expected Utility Maximization

Mean/variance choice under uncertainty is justified when risks are small. When risks are small, one can approximate the utility function well by quadratic utility, as possible outcomes fall within a narrow range. The expected value of a quadratic is a function of mean and variance, so mean/variance choice is equivalent to expected utility maximization.

Small Risks via Stochastic Calculus

Let us analyze expected utility maximization among small risks by stochastic calculus.

We show a strong result: for small risks, expected utility maximization is equivalent to maximizing a *linear* function of the mean and the variance.

Risk Aversion

The tradeoff between mean and variance depends on the risk aversion of the individual. When risk aversion is high, a small increase in variance constitutes a large drop in expected utility. Conversely, when risk aversion is low, even a large increase in variance constitutes only a small drop in expected utility.

Absolute and Relative Risk Aversion

Arrow [1] puts forward two measures of the degree of risk aversion.

Definition 1 (Absolute Risk Aversion) *The absolute risk aversion is*

$$-u''/u' > 0.$$

Definition 2 (Relative Risk Aversion) *The relative risk aversion is*

$$-wu''/u' > 0.$$

The absolute and relative risk aversion are both invariant to positive linear transformation of utility.

Interpretation

The absolute risk aversion shows the willingness to take a risk of a given absolute size. For example, how willing is the individual to risk $100?

The relative risk aversion shows the willingness to take a risk of a given size relative to wealth. For example, how willing is the individual to risk 10 *per cent* of his wealth?

The absolute and the relative risk aversion set the tradeoff between mean and variance.

If the mean and variance are expressed as absolute dollar amounts, then the absolute risk aversion sets the tradeoff.

If the mean and variance are expressed as amounts relative to initial wealth, then the relative risk aversion sets the tradeoff.

Small Risk

Consider a small risk, for which the mean and the variance are expressed as absolute dollar amounts. The probability distribution of wealth is

$$w \sim \mathrm{N}(\widetilde{w} + m\Delta t, s^2\Delta t).$$

Base wealth $\widetilde{w}$ is fixed. Here Δt is a small number; as it changes, the mean and the variance both change in proportion. We write

$$w = \widetilde{w} + m\Delta t + s\Delta z, \tag{1}$$

in which $\Delta z \sim \mathrm{N}(0, \Delta t)$. The issue is to study how expected utility depends on m and s^2.

Small Risk via Stochastic Calculus

Applying stochastic calculus, consider the limit as Δt approaches the infinitesimal $\mathrm{d}t$. The limit of (1) as

$$w_{t+\mathrm{d}t} \equiv w_t + m\,\mathrm{d}t + s\,\mathrm{d}z.$$

Wealth is $w_{t+\mathrm{d}t}$, and base wealth is w_t; define the change in wealth $\mathrm{d}w = w_{t+\mathrm{d}t} - w_t$. Here z is Wiener-Brownian motion, $\mathrm{d}z \equiv z_{t+\mathrm{d}t} - z_t$, $\mathrm{d}z \sim \mathrm{N}(0, \mathrm{d}t)$. Thus wealth is distributed

$$w_{t+\mathrm{d}t} \sim \mathrm{N}(w_t + m\,\mathrm{d}t, s^2\,\mathrm{d}t).$$

Because $\mathrm{d}t$ is infinitesimal, the risk is small.

Expected Utility

The individual maximizes expected utility

$$\mathrm{E}[u(w_{t+\mathrm{d}t})].$$

The issue is to study how expected utility depends on m and s^2.

Quadratic Utility

By Itô's formula,

$$u(w_{t+\mathrm{d}t}) = u(w_t) + u'(w_t)\,\mathrm{d}w + \frac{1}{2}u''(w_t)(\mathrm{d}w)^2.$$

The utility is quadratic in dw, so the mean m and the variance s^2 determine the expected utility.

Evaluation of Expected Utility

We evaluate

$$
\begin{aligned}
u(w_{t+\mathrm{d}t}) &= u + u' \,\mathrm{d}w + \frac{1}{2}u''(\mathrm{d}w)^2 \\
&= u + u'(m\,\mathrm{d}t + s\,\mathrm{d}z) + \frac{1}{2}u''(m\,\mathrm{d}t + s\,\mathrm{d}z)^2 \\
&= u + \left(u'm + \frac{1}{2}u''s^2\right)\mathrm{d}t + u's\,\mathrm{d}z.
\end{aligned}
$$

Here the only stochastic term is dz, which has expected value zero.

Expected utility is

$$\mathrm{E}[u(w_{t+\mathrm{d}t})] = u + \left(u'm + \frac{1}{2}u''s^2\right)\mathrm{d}t$$
$$= u + u'\left[m - \frac{1}{2}\left(-\frac{u''}{u'}\right)s^2\right]\mathrm{d}t.$$

Expected utility is therefore determined by the expression in brackets, a linear function of the mean m and the variance s^2,

$$m - \frac{1}{2}\left(-\frac{u''}{u'}\right)s^2.$$

A higher mean raises expected utility and a higher variance lowers expected utility. Because the mean and the variance are expressed as absolute dollar amounts, the absolute risk aversion sets the tradeoff.

Small Risk Relative to Wealth

Consider the expected utility for a risk of a given size relative to wealth. Define

$$w_{t+\mathrm{d}t} \equiv w_t(1+m\,\mathrm{d}t+s\,\mathrm{d}z).$$

Now m and s have the interpretation as the mean and the standard deviation relative to wealth.

$$
\begin{aligned}
u(w_{t+\mathrm{d}t}) &= u + u'\,\mathrm{d}w + \frac{1}{2}u''(\mathrm{d}w)^2 \\
&= u + u'w(m\,\mathrm{d}t + s\,\mathrm{d}z) + \frac{1}{2}u''w^2(m\,\mathrm{d}t + s\,\mathrm{d}z)^2 \\
&= u + \left(u'wm + \frac{1}{2}u''w^2s^2\right)\mathrm{d}t + u'ws\,\mathrm{d}z.
\end{aligned}
$$

Here the only stochastic term is dz, which has expected value zero.

Expected Utility

Expected utility is

$$\begin{aligned} \mathrm{E}[u(w_{t+\mathrm{d}t})] &= u + \left(u' w m + \frac{1}{2} u'' w^2 s^2 \right) \mathrm{d}t \\ &= u + u' w \left[m - \frac{1}{2} \left(-\frac{w u''}{u'} \right) s^2 \right] \mathrm{d}t . \end{aligned}$$

Expected utility is determined by the expression in brackets, a linear function of the mean m and the variance s^2,

$$m - \frac{1}{2}\left(-\frac{wu''}{u'}\right)s^2.$$

A higher mean raises expected utility, and a higher variance lowers expected utility. The relative risk aversion shows the tradeoff between the mean and the variance, where these two magnitudes now describe the risk relative to wealth.

References

[1] Kenneth J. Arrow. The theory of risk aversion. In *Individual Choice under Certainty and Uncertainty, collected papers of Kenneth J. Arrow*, pages 147–171. Harvard University Press, Cambridge, MA, 1984. HD30.23A74 1984.