Portfolio Choice with Many Risky Assets

Consider portfolio choice with many risky assets and a risk-free asset. It turns out that the single risky asset model still applies!

Efficient Frontier for Risky Assets

Consider all possible portfolios of risky assets only. These portfolios determine all possible combinations of mean and standard deviation. In (standard deviation, mean)-space, typically a parabola forms the boundary of these combinations.

The upper left boundary constitutes the *efficient frontier for risky assets*. For any point on the frontier, there is some portfolio that achieves this combination of mean and standard deviation, and no other portfolio can achieve either the same mean with a lower standard deviation, or a lower standard deviation with the same mean. One refers to this portfolio as *efficient*.

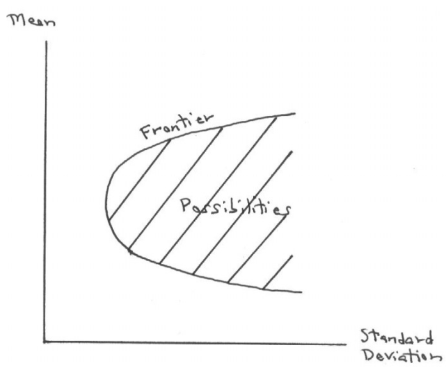

Figure 1: Efficient Frontier for Risky Assets

Risky Portfolio and Risk-Free Portfolio

Consider a portfolio in which one invests the fraction f of wealth in a portfolio of risky assets and the fraction $1-f$ in the risk-free asset. In (standard deviation, mean)-space, the possible combinations lie along the straight line connecting the (standard deviation, mean)-values for each.

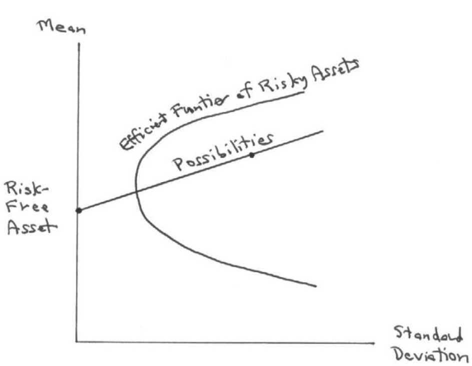

Figure 2: Possibilities

Efficient Frontier for All Assets

The *efficient frontier for all assets* is the line tangent to the efficient frontier of risky assets passing though the point $(0, \text{risk-free rate of return})$ for the risk-free asset.

Definition 1 (Efficient Portfolio of Risky Assets) *The efficient portfolio of risky assets is the portfolio corresponding to the tangent point.*

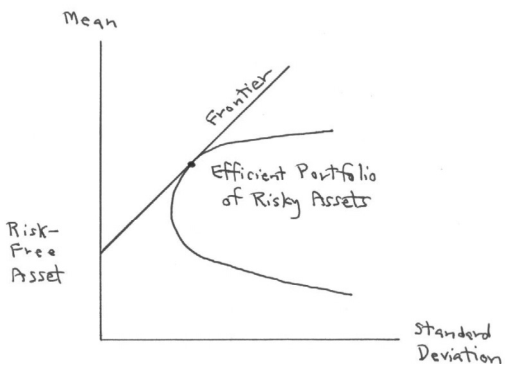

Figure 3: Separation Theorem, Stage 1

Reduction to Two-Asset Model

The optimum portfolio choice reduces to the two-asset model with a single risky asset and a risk-free asset: invest the fraction f of wealth in the efficient portfolio of risky assets and the fraction $1-f$ in the risk-free asset, in which

$$f = \frac{\mu}{\alpha\sigma^2}.$$

Here μ is the mean excess return on the efficient portfolio of risky assets, σ is the standard deviation, and α is the relative risk aversion.

Separation Theorem

Theorem 2 *(Tobin [1]) Portfolio choice is separated into two stages:*

- *Find the efficient portfolio of risky assets;*
- *Find the optimum fraction to invest in the efficient portfolio of risky assets and the risk-free asset.*

The role of risk aversion is confined to the second stage and plays no role in the first stage.

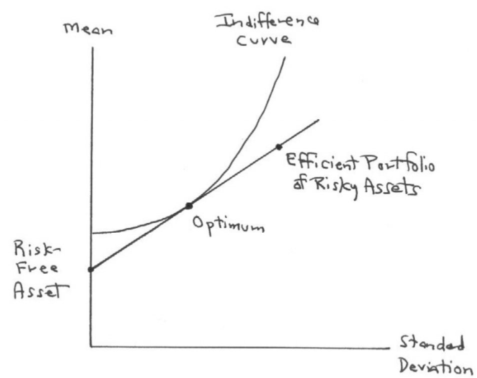

Figure 4: Separation Theorem, Stage 2

Tangent Slope

Suppose that $\boldsymbol{m}$ is the vector of mean excess returns on the risky assets and that V is the variance (non-singular). Let $\boldsymbol{f}$ denote a portfolio of risky assets, in which $\boldsymbol{f}_i$ is the fraction of wealth invested in asset i, normalized so

$$\boldsymbol{f}^\top \mathbf{1} = 1 \tag{1}$$

($\mathbf{1}$ is a vector with every component one).

The tangent line defining the efficient portfolio of risky assets maximizes the slope,

$$\frac{\boldsymbol{f}^\top \boldsymbol{m}}{\sqrt{\boldsymbol{f}^\top V \boldsymbol{f}}}, \tag{2}$$

the ratio of the mean excess return to the standard deviation.

Maximum Slope

The normalization $\boldsymbol{f}^\top \mathbf{1} = 1$ notwithstanding, the ratio (2) is unaffected by multiplying $\boldsymbol{f}$ by a positive number. Consequently one can calculate $\boldsymbol{f}$ by solving

$$\max_{\boldsymbol{f}} \boldsymbol{f}^\top \boldsymbol{m}$$

such that

$$\boldsymbol{f}^\top V \boldsymbol{f} = 1.$$

Define the Lagrangian

$$L = \boldsymbol{f}^\top \boldsymbol{m} + \frac{\lambda}{2}\left(1 - \boldsymbol{f}^\top V \boldsymbol{f}\right),$$

in which λ is a Lagrange multiplier. The first-order conditions for a maximum are

$$0 = \frac{\partial L}{\partial \boldsymbol{f}} = \boldsymbol{m} - \lambda V \boldsymbol{f}$$

$$0 = \frac{\partial L}{\partial \lambda} = \frac{1}{2}\left(1 - \boldsymbol{f}^\top V \boldsymbol{f}\right).$$

From the first equation,

$$\boldsymbol{f} = \frac{1}{\lambda} V^{-1} \boldsymbol{m}.$$

Invoking the normalization (1) gives the following theorem.

Theorem 3 (Efficient Portfolio of Risky Assets) *The efficient portfolio of risky assets is*

$$\boldsymbol{f} = \frac{V^{-1}\boldsymbol{m}}{\mathbf{1}^\top V^{-1} \boldsymbol{m}}.$$

Optimum Portfolio Choice

The mean excess return on the efficient portfolio of risky assets is

$$\mu = \boldsymbol{f}^\top \boldsymbol{m} = \frac{\boldsymbol{m}^\top V^{-1} \boldsymbol{m}}{\mathbf{1}^\top V^{-1} \boldsymbol{m}},$$

and the variance is

$$\sigma^2 = \boldsymbol{f}^\top V \boldsymbol{f} = \left(\frac{\boldsymbol{m}^\top V^{-1}}{\mathbf{1}^\top V^{-1} \boldsymbol{m}}\right) V \left(\frac{V^{-1}\boldsymbol{m}}{\mathbf{1}^\top V^{-1} \boldsymbol{m}}\right) = \frac{\boldsymbol{m}^\top V^{-1} \boldsymbol{m}}{\left(\mathbf{1}^\top V^{-1} \boldsymbol{m}\right)^2}.$$

The fraction of wealth invested in the efficient portfolio of risky assets is therefore

$$\frac{\mu}{\alpha\sigma^2} = \frac{1}{\alpha} \frac{\left(\frac{\boldsymbol{m}^\top V^{-1} \boldsymbol{m}}{\mathbf{1}^\top V^{-1} \boldsymbol{m}}\right)}{\left[\frac{\boldsymbol{m}^\top V^{-1} \boldsymbol{m}}{\left(\mathbf{1}^\top V^{-1} \boldsymbol{m}\right)^2}\right]} = \frac{1}{\alpha} \mathbf{1}^\top V^{-1} \boldsymbol{m}.$$

Hence the fraction of wealth invested in the risky assets is

$$\left(\frac{\mu}{\alpha\sigma^2}\right)\boldsymbol{f} = \frac{1}{\alpha} \mathbf{1}^\top V^{-1} \boldsymbol{m} \left(\frac{V^{-1}\boldsymbol{m}}{\mathbf{1}^\top V^{-1} \boldsymbol{m}}\right) = \frac{1}{\alpha} V^{-1} \boldsymbol{m}, \qquad (3)$$

and the fraction of wealth invested in the risk-free asset is

$$1 - \frac{1}{\alpha} \mathbf{1}^\top V^{-1} \boldsymbol{m}.$$

Expected Utility Maximization

Alternatively, one can maximize expected utility directly, a quicker but less intuitive derivation.

Define the vector $\boldsymbol{f}$ as the fraction of wealth invested in the risky assets, and $1-\mathbf{1}^{\top}\boldsymbol{f}$ is the fraction of wealth invested in the risk-free asset. (Note the change in notation.)

The mean rate of return on the portfolio is

$$\mu = \boldsymbol{f}^{\top}(\boldsymbol{m}+r\mathbf{1}) + \left(1-\mathbf{1}^{\top}\boldsymbol{f}\right) r = r + \boldsymbol{f}^{\top}\boldsymbol{m},$$

and the variance is

$$\sigma^2 = \boldsymbol{f}^{\top} V \boldsymbol{f}.$$

Expected utility is

$$\mu - \frac{\alpha}{2}\sigma^2 = \left(r + \boldsymbol{f}^{\top}\boldsymbol{m}\right) - \frac{\alpha}{2}\boldsymbol{f}^{\top} V \boldsymbol{f}.$$

The first-order condition for expected utility maximization is

$$0 = \frac{\partial \cdot}{\partial f} = \boldsymbol{m} - \alpha V \boldsymbol{f},$$

so

$$\boldsymbol{f} = \frac{1}{\alpha} V^{-1}\boldsymbol{m},$$

in agreement with (3).

References

[1] James Tobin. Liquidity preference as behavior towards risk. *Review of Economic Studies*, XXV(2):65–86, February 1958. HB1R4.