Runs Test

A simple statistical test of the random-walk theory is a *runs test*. For daily data, a run is defined as a sequence of days in which the stock price changes in the same direction.

For example, consider the following combination of upward and downward price changes:

$$+ + - - + - + - - - + + .$$

A $+$ sign means that the stock price increased, and a $-$ sign means that the stock price decreased. Thus the example has 7 runs, in 12 observations.

Expected Number of Runs

Suppose that the random-walk theory holds: each day there is a 50% chance of an increase in the price and a 50% chance of a decrease.

For n observations, what is the expected number of runs?

The expected number of runs is

$$\frac{n}{2}.$$

Each day the probability that a new run starts is one half, and the probability that the current run continues is one half.

(More precisely, the expected number of runs is

$$1+\frac{n-1}{2}=\frac{n+1}{2},$$

since the first day necessarily starts a new run.)

Momentum

Momentum investing rejects the random-walk theory. The assumption is that trends continue: a price increase implies further price increases; a price decrease implies further price decreases. One buys when the stock price is rising and sells when it is falling.

According to the momentum theory, runs tend to continue.
Hence the expected number of runs is less.

One-Tailed Test

It is natural to test the random-walk theory against the momentum theory. A one-tailed test is natural, as the momentum theory predicts fewer runs than the random-walk theory.

Critical Value

If the random-walk theory is true, the expected number of runs is $n/2$, and the standard deviation of the number of runs is

$$\frac{\sqrt{n}}{2}.$$

With probability 5%, the number of runs will lie more than 1.64 standard deviations below the expected value, and this number is the critical value.

Example

For $n = 400$, then

$$1.64 \frac{\sqrt{n}}{2} = 16.4.$$

The expected number of runs is 200.

Hence one rejects the null hypothesis that the random-walk theory is true if the number of runs is 183 or less; this low number could occur by chance only 5% of the time.

If the number of runs is 184 or more, than one accepts the null hypothesis. This number is close enough to 200 to be compatible with the random-walk theory.

Technical Note

A possibility is that on certain days the stock price does not change. One can deal with this possibility just by ignoring the observations on these days.

Too Many Long Runs?

Some observers think that too many long runs occur, too many for the random-walk theory to be true.

Runs Probability

The table shows the probability of runs of different lengths; of course the probabilities sum to one.

Length	Probability
1	$\frac{1}{2}$
2	$\frac{1}{4}$
3	$\frac{1}{8}$
...	

As the length rises by one, the probability falls in half, since there is a 50% chance that the run ends.

Average Length of a Run

The mean length of a run is therefore

$$\begin{aligned}\left(1\times\frac{1}{2}\right)+\left(2\times\frac{1}{4}\right)+\left(3\times\frac{1}{8}\right)+\cdots & \\ &= \left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots\right)+\left(\frac{1}{4}+\frac{1}{8}+\cdots\right) \\ &\quad +\left(\frac{1}{8}+\cdots\right)+\cdots \\ &= 1+\frac{1}{2}+\frac{1}{4}+\cdots \\ &= 2.\end{aligned}$$

This result agrees with what we knew already: each day there is a 50% chance of starting a new run, so the mean length of a run must be

$$\frac{1}{\frac{1}{2}} = 2.$$

Probability of a Run

The unconditional probability per day of a run of length n or more is therefore

$$\frac{1}{2^n}.$$

If $256 = 2^8$ were the number of business days per year, then the unconditional mean number of runs of length n or more per year would be

$$\frac{1}{2^n} \times 2^8 = 2^{8-n}.$$

Long Runs of the Dow Jones Industrial Average

During the period 1896-2006, the number of runs of 11 days or more was 8. According to the random-walk model, the expected number was

$$2^{8-11} \times 111 \approx 14,$$

so the number of long runs was lower than expected!

The longest run was 14 days, whereas the expected number of runs of 14 days or more was

$$2^{8-14} \times 111 \approx 1.7,$$

a reasonable agreement.

Probability Not One Half

Since stock prices tend to rise in the long run, the probability of a price increase each day must in fact be slightly more than one half. Let

$$\frac{1}{2}+x$$

denote this probability, so

$$\frac{1}{2}-x$$

is the probability of a price decline.

Then the unconditional probability of starting a new run on a given day is

$$\left(\frac{1}{2}+x\right)\left(\frac{1}{2}-x\right)+\left(\frac{1}{2}-x\right)\left(\frac{1}{2}+x\right)=\frac{1}{2}-2x^2.$$

Since the effect of x is second-order, the probability is nearly one half, as long as x is not too large.