Excess Return

Let r denote the risk-free rate of return.

Let $\mathrm{d}x_i$ denote the excess return on asset i: one dollar invested at time t is worth $1 + r\,\mathrm{d}t + \mathrm{d}x_i$ at time $t + \mathrm{d}t$.

Let $\mathrm{d}x_m$ denote the excess return on the market portfolio of all assets.

Risk Premium on the Market Portfolio

In the capital-asset pricing model, the risk premium on the market portfolio is the relative risk aversion of a typical investor times the variance of the return on the market portfolio,

$$\mathrm{E}\left(\mathrm{d}x_m\right) = \alpha \,\mathrm{Var}\left(\mathrm{d}x_m\right).$$

Beta Coefficient

The risk premium on an asset is its beta coefficient times the risk premium on the market portfolio,

$$\mathrm{E}\left(\mathrm{d}x_i\right) = \beta_i \, \mathrm{E}\left(\mathrm{d}x_m\right).$$

Risk Premium as Covariance

Alternatively, one can express the risk premium via the covariance of the return with the return on the market portfolio. The beta coefficient is the coefficient in the least-squares linear regression of the asset return on the return on the market portfolio,

$$\beta_i = \frac{\mathrm{Cov}\left(\mathrm{d}x_i, \mathrm{d}x_m\right)}{\mathrm{Var}\left(\mathrm{d}x_m\right)}.$$

Consequently the risk premium on an asset is the relative risk aversion times the covariance of its return with the return on the market portfolio:

$$\begin{aligned} \mathrm{E}\left(\mathrm{d}x_i\right) &= \beta_i \,\mathrm{E}\left(\mathrm{d}x_m\right) \\ &= \left[\frac{\mathrm{Cov}\left(\mathrm{d}x_i, \mathrm{d}x_m\right)}{\mathrm{Var}\left(\mathrm{d}x_m\right)}\right]\left[\alpha \,\mathrm{Var}\left(\mathrm{d}x_m\right)\right] \\ &= \alpha \,\mathrm{Cov}\left(\mathrm{d}x_i, \mathrm{d}x_m\right). \end{aligned}$$

Theorem 1 (Risk Premium as Covariance)

$$\mathrm{E}(\mathrm{d}x_i) = \alpha \, \mathrm{Cov}(\mathrm{d}x_i, \mathrm{d}x_m) \,. \tag{1}$$

Portfolio Choice

Our earlier formula for optimum portfolio choice provides an alternate derivation. If the vector $\mathrm{d}\boldsymbol{x}$ of excess returns on risky assets has the probability distribution

$$\mathrm{d}\boldsymbol{x} \sim \mathrm{N}\left(\boldsymbol{m}\,\mathrm{d}t, V\,\mathrm{d}t\right),$$

with V nonsingular, then the optimum portfolio choice is

$$\boldsymbol{f} = \frac{1}{\alpha} V^{-1} \boldsymbol{m}. \tag{2}$$

Here the vector $\boldsymbol{f}$ denotes the fraction of wealth invested in the various risky assets. The remaining fraction $1 - \mathbf{1}'\boldsymbol{f}$ is invested in the risk-free asset.

For demand to equal supply, this $\boldsymbol{f}$ must also be the fraction of the market portfolio for the various risky assets. The excess return on the market portfolio is therefore

$$\mathrm{d}x_m = \boldsymbol{f}' \mathrm{d}\boldsymbol{x},$$

and

$$\mathrm{Cov}\left(\mathrm{d}\boldsymbol{x}, \mathrm{d}x_m\right) = \mathrm{d}\boldsymbol{x}\left(\mathrm{d}\boldsymbol{x}\right)^{\top} \boldsymbol{f} = V\boldsymbol{f}\,\mathrm{d}t\,.$$

By (2),

$$\boldsymbol{m}\,\mathrm{d}t = \alpha V\boldsymbol{f}\,\mathrm{d}t = \alpha\,\mathrm{Cov}\left(\mathrm{d}\boldsymbol{x}, \mathrm{d}x_m\right),$$

the vector form of (1).

First-Order Condition

The first-order condition for optimum portfolio choice provides yet another derivation.

If $\mathrm{d}a$ is the return on the optimum portfolio and $\mathrm{d}a_i$ is the return on asset i, then

$$0 = \mathrm{E}\left[w_t u'\left(w_{t+\mathrm{d}t}\right)\left(\mathrm{d}a_i - \mathrm{d}a\right)\right]$$

is a first-order condition for optimum portfolio choice.

If utility $u\left(w_{t+\mathrm{d}t}\right)$ depends only on wealth and is not state dependent, then the first-order condition is also

$$0 = \left[\mathrm{E}\left(\mathrm{d}a_i\right) - \mathrm{E}\left(\mathrm{d}a\right)\right] - \alpha\,\mathrm{d}a\left(\mathrm{d}a - \mathrm{d}a_i\right). \tag{3}$$

Risk Premium on the Market Portfolio

A typical investor buys the market portfolio, so for him $\mathrm{d}a$ is the return on the market portfolio. Taking asset i as the risk-free asset, with return $\mathrm{d}a_i = r\,\mathrm{d}t$, then (3) implies

$$\mathrm{E}\left(\mathrm{d}a\right) = r\,\mathrm{d}t + \alpha\left(\mathrm{d}a\right)^2. \tag{4}$$

The risk premium on the market portfolio is the relative risk aversion times the variance of its return.

Risk Premium on an Asset

For an arbitrary asset i, substituting (4) into (3) then yields

$$\begin{aligned} \mathrm{E}\left(\mathrm{d}a_i\right) &= \mathrm{E}\left(\mathrm{d}a\right) + \alpha\,\mathrm{d}a\left(\mathrm{d}a_i - \mathrm{d}a\right) \\ &= \left[\mathrm{E}\left(\mathrm{d}a\right) - \alpha\left(\mathrm{d}a\right)^2\right] + \alpha\,\mathrm{d}a_i\,\mathrm{d}a \\ &= r\,\mathrm{d}t + \alpha\,\mathrm{d}a_i\,\mathrm{d}a. \end{aligned}$$

The risk premium is the relative risk aversion times the covariance of the return with the return on the market.