Notation

Let P_t^τ denote the price at time t of a risk-free pure discount bond worth one dollar at its maturity in τ years, at time $t+\tau$. Thus $P_t^0 = 1$.

Let R_t^τ denote the yield to maturity on this bond. By definition,

$$P_t^\tau = \mathrm{e}^{-\tau R_t^\tau}.$$

Expectations Theory

The expectations theory says that the long-term interest rate is the average of current and expected future short-term rates. For example, the two-year interest rate is the average of the current one-year rate and the one-year rate expected for next year,

$$R_t^2 = \frac{1}{2}\left[R_t^1 + \mathrm{E}_t\left(R_{t+1}^1\right)\right]. \tag{1}$$

Perfect Foresight

For the case of perfect foresight, the expectations theory follows from the rate-of-return condition, that long-term and short-term bonds must have equal holding-period returns, regardless of the holding period.

One-Year Time Horizon

Consider a one-year time horizon. One dollar invested at time t in a one-year bond is worth

$$\frac{1}{P_t^1}$$

at time $t+1$. One dollar will buy $1/P_t^2$ units of a two-year bond at time t, so the investment will be worth

$$\frac{P_{t+1}^1}{P_t^2}$$

at time $t+1$.

The rate-of-return condition says that the two returns must be equal,

$$\frac{1}{P_t^1} = \frac{P_{t+1}^1}{P_t^2}. \tag{2}$$

The expectations theory (1) follows:

$$\begin{aligned} R_t^2 &= -\frac{1}{2}\ln P_t^2 \\ &= -\frac{1}{2}\left[\ln P_t^1 + \ln P_{t+1}^1\right], \text{ by (2)} \\ &= \frac{1}{2}\left(R_t^1 + R_{t+1}^1\right). \end{aligned}$$

Two-Year Time Horizon

Alternatively, consider a two-year time horizon. One dollar invested at time t in a two-year bond is worth

$$\frac{1}{P_t^2}$$

at time $t+2$. One dollar invested in a one-year bond at time t and then rolled over into another one-year bond at time $t+1$ will be worth

$$\frac{1}{P_t^1 P_{t+1}^1}$$

at time $t+2$.

The rate-of-return condition says that the two returns must be equal,

$$\frac{1}{P_t^1 P_{t+1}^1} = \frac{1}{P_t^2}. \tag{3}$$

Since (3) is the same as (2), again the expectations theory (1) follows.

Interest-Rate Risk

Alternatively, suppose that there is uncertainty about future interest rates. At time t, P_{t+1}^1 and R_{t+1}^1 are uncertain.

One-Year Time Horizon

One might require risk neutrality with a one-year time horizon. The condition that both investments have the same expected rate of return is

$$\frac{1}{P_t^1} = \frac{\mathrm{E}_t\left(P_{t+1}^1\right)}{P_t^2}. \tag{4}$$

Two-Year Time Horizon

Alternatively, one might require risk neutrality with a two-year time horizon. The condition that both investments have the same expected rate of return is

$$\frac{1}{P_t^1} \mathrm{E}_t \left(\frac{1}{P_{t+1}^1} \right) = \frac{1}{P_t^2}. \tag{5}$$

Inconsistency

With uncertainty, the risk-neutrality conditions (4) and (5) are inconsistent (Stiglitz [1]). Both cannot be true simultaneously: by Jensen's inequality (1),

$$\mathrm{E}_t\left(\frac{1}{P_{t+1}^1}\right) > \frac{1}{\mathrm{E}_t\left(P_{t+1}^1\right)}.$$

Autocorrelation

The source of the inconsistency is the autocorrelation of the holding-period return on a bond. The yield to maturity fixes the total return over the life of the bond. Consequently a higher holding-period return in one period is necessarily offset by a lower holding-period return in a later period.

Expectations Theory

Furthermore, both risk-neutrality conditions are inconsistent with the expectations theory (1).

One-Year Time Horizon

Risk-neutrality with a one-year time horizon violates the expectations theory:

$$
\begin{aligned}
R_t^2 &= -\frac{1}{2} \ln P_t^2 \\
&= -\frac{1}{2} \left[\ln P_t^1 + \ln \mathrm{E}_t \left(P_{t+1}^1 \right) \right], \text{ by (4)} \\
&< -\frac{1}{2} \left[\ln P_t^1 + \mathrm{E}_t \left(\ln P_{t+1}^1 \right) \right], \text{ by proposition (1)} \\
&= \frac{1}{2} \left[R_t^1 + \mathrm{E}_t \left(R_{t+1}^1 \right) \right].
\end{aligned}
$$

Two-Year Time Horizon

Risk-neutrality with a two-year time horizon also violates the expectations theory, but in the opposite direction:

$$\begin{aligned}
R_t^2 &= -\frac{1}{2}\ln P_t^2 \\
&= -\frac{1}{2}\left[\ln P_t^1 - \ln \mathrm{E}_t\left(\frac{1}{P_{t+1}^1}\right)\right], \text{ by (5)} \\
&> -\frac{1}{2}\left[\ln P_t^1 - \mathrm{E}_t\left(\ln\frac{1}{P_{t+1}^1}\right)\right], \text{ by proposition (1)} \\
&= \frac{1}{2}\left[R_t^1 + \mathrm{E}_t\left(R_{t+1}^1\right)\right].
\end{aligned}$$

Implication of the Expectations Theory

Conversely, one can work backwards from the expectations theory (1) to find its implication for the risk-neutrality conditions (4) and (5).

One-Year Time Horizon

For the one-year time horizon, the expected holding-period return on the two-year bond exceeds the expected return on the one-year bond:

$$
\begin{aligned}
\frac{\mathrm{E}_t\left(P_{t+1}^1\right)}{P_t^2} &= \mathrm{e}^{2R_t^2}\,\mathrm{E}_t\left(\mathrm{e}^{-R_{t+1}^1}\right) \\
&> \mathrm{e}^{2R_t^2}\,\mathrm{e}^{-\mathrm{E}_t\left(R_{t+1}^1\right)}, \text{ by proposition (1)} \\
&= \mathrm{e}^{R_t^1}, \text{ by (1)} \\
&= \frac{1}{P_t^1}.
\end{aligned}
$$

Two-Year Time Horizon

For the two-year time horizon, the expected return on the sequence of one-year bonds exceeds the expected return on the two-year bond:

$$\begin{aligned}
\frac{1}{P_t^1} \mathrm{E}_t \left(\frac{1}{P_{t+1}^1} \right) &= \mathrm{e}^{R_t^1} \mathrm{E}_t \left(\mathrm{e}^{R_{t+1}^1} \right) \\
&> \mathrm{e}^{R_t^1} \mathrm{e}^{\mathrm{E}_t \left(R_{t+1}^1 \right)}, \text{ by proposition (1)} \\
&= \mathrm{e}^{2R_t^2}, \text{ by (1)} \\
&= \frac{1}{P_t^2}.
\end{aligned}$$

Approximation

One interpretation is to see the expectations theory just as an approximation.

Even though there is no satisfactory theoretical derivation of the theory, one can nevertheless test the theory with data.

Jensen's Inequality

Proposition 1 (Jensen's Inequality) *For a strictly convex function f of a random variable x, then*

$$\mathrm{E}\left[f(x)\right] > f\left[\mathrm{E}(x)\right],$$

as long as x is uncertain.

The inequality is reversed if the function is strictly concave.

The inverse $1/x$ and the exponential e^x are strictly convex, and the logarithm ln is strictly concave.

References

[1] Joseph E. Stiglitz. A consumption-oriented theory of the demand for financial assets and the term structure of interest rates. *Review of Economic Studies*, 37(3):321–351, July 1970.