Payoff Space

The set of possible payoffs is the range $\mathrm{R}(A)$. This payoff space is a subspace of the state space and is a Euclidean space in its own right.

Pricing Kernel

By the law of one price, two portfolios with the same payoff $\boldsymbol{y}$ have the same cost. Cost is linear, so there exists some linear function $\langle \boldsymbol{p}, \boldsymbol{y} \rangle$ on the payoff space that equals the cost of the portfolio. Necessarily $\boldsymbol{p} \in \mathrm{R}(A)$ is unique.

For a portfolio $\boldsymbol{x}$ such that $\boldsymbol{y} = A\boldsymbol{x}$, the cost

$$\langle \boldsymbol{v}, \boldsymbol{x} \rangle = \langle \boldsymbol{p}, \boldsymbol{y} \rangle = \langle \boldsymbol{p}, A\boldsymbol{x} \rangle ,$$

so $\boldsymbol{p}$ is a stochastic discount factor. One refers to $\boldsymbol{p}$ as the *pricing kernel*.

Definition 1 (Pricing Kernel) *The pricing kernel is the unique stochastic discount factor in the payoff space.*

Formula for the Pricing Kernel

For any stochastic discount factor,

$$A^{\top} \boldsymbol{y} = \boldsymbol{v}.$$

There exists a solution $\boldsymbol{y}$ if and only if the law of one price holds. The solution set is then

$$\left\{ A^{\top +} \boldsymbol{v} \right\} + \mathrm{N}\left(A^{\top} \right) . \tag{1}$$

By the fundamental theorem of linear algebra, $\mathrm{N}\left(A^{\top}\right)$ and the payoff space $\mathrm{R}(A)$ are orthogonal, so the pricing kernel must be the expression in braces.

Theorem 2 (Pricing Kernel) *The pricing kernel is*

$$\boldsymbol{p} = A^{\top +} \boldsymbol{v}. \tag{2}$$

Even if there is no opportunity for profitable arbitrage, it is not necessary that $\boldsymbol{p} \gg \boldsymbol{0}$.

Least-Squares Interpretation

Consider the least-squares linear regression of the dependent variable $\boldsymbol{v}$ on the independent variables $A^{\top}$ with regression coefficients $\boldsymbol{p}$.

Problem 3 (Pricing Kernel as Least Squares)

$$\min_{\boldsymbol{p}} \left\langle \boldsymbol{v} - A^{\top} \boldsymbol{p}, \boldsymbol{v} - A^{\top} \boldsymbol{p} \right\rangle .$$

The solution set for the regression coefficients is (1), and (2) is the unique solution in $\mathrm{R}(A)$.

Expectations Kernel

The expected payoff on portfolio $\boldsymbol{x}$ is $\langle \mathbf{1}, A\boldsymbol{x} \rangle$.

Since expectation is linear, the expected payoff is some linear function $\langle \boldsymbol{e}, \boldsymbol{y} \rangle$ on the payoff space, for a unique $\boldsymbol{e}$.

One refers to $\boldsymbol{e}$ as the *expectations kernel*.

If there exists a risk-free portfolio (if $\mathbf{1} \in \mathrm{R}(A)$), then of course $\boldsymbol{e} = \mathbf{1}$.

If there is no risk-free portfolio, necessarily $\boldsymbol{e} \neq \mathbf{1}$.

For any $\boldsymbol{x}$, $\boldsymbol{e}$ must satisfy

$$\langle \mathbf{1}, A\boldsymbol{x} \rangle = \langle \boldsymbol{e}, A\boldsymbol{x} \rangle,$$

so

$$\left\langle A^\top \mathbf{1}, \boldsymbol{x} \right\rangle = \left\langle A^\top \boldsymbol{e}, \boldsymbol{x} \right\rangle.$$

Since this condition must hold for any $\boldsymbol{x}$, therefore

$$A^\top \boldsymbol{e} = A^\top \mathbf{1}. \tag{3}$$

Since $\boldsymbol{e}$ necessarily exists, this equation has a solution. The solution set is

$$\left\{ A^{\top+} A^\top \mathbf{1} \right\} + \mathrm{N}\left(A^\top \right).$$

By the fundamental theorem of linear algebra, $\mathrm{N}\left(A^\top\right)$ is orthogonal to the payoff space $\mathrm{R}(A)$, so $\boldsymbol{e}$ must be the expression in braces.

Theorem 4 (Expectations Kernel)

$$\boldsymbol{e} = A^{\top+} A^\top \mathbf{1}.$$

Least-Squares Interpretation

One can interpret the expectations kernel as the solution to a least-squares problem.

Problem 5 (Expectations Kernel as Least Squares)

$$\min_{\boldsymbol{e} \in \mathrm{R}(A)} \langle \mathbf{1} - \boldsymbol{e}, \mathbf{1} - \boldsymbol{e} \rangle.$$

Interpret $\boldsymbol{e}$ as the fitted values of the least-squares linear regression of the dependent variable $\mathbf{1}$ on the dependent variables A. That $\boldsymbol{e} \in \mathrm{R}(A)$ means that $\boldsymbol{e} = A\boldsymbol{x}$ for regression coefficients $\boldsymbol{x}$.

The condition (3) says that the residual $\mathbf{1}-\boldsymbol{e}$ is orthogonal to the dependent variables. Hence the necessary and sufficient conditions for least-squares linear regression are fulfilled. The expectations kernel (3) is the unique solution to least-squares problem (5).

Sum of Squares

Since the explained sum of squares $\langle \boldsymbol{e},\boldsymbol{e}\rangle$ in (5) is necessarily less than or equal to the total sum of squares $\langle \mathbf{1},\mathbf{1}\rangle = 1$, therefore

$$\langle \boldsymbol{e},\boldsymbol{e}\rangle \leq 1,$$

with equality only if $\boldsymbol{e}=\mathbf{1}$.

Complete Markets

The asset market is complete if the payoff space is the state space; any payoff in the state space can be attained by some portfolio.

Since A is onto, $A^{+}=A^{\top}\left(AA^{\top}\right)^{-1}$.

The pricing kernel is then

$$\boldsymbol{p}=A^{\top+}\boldsymbol{v}=\left(AA^{\top}\right)^{-1}A\boldsymbol{v},$$

in accord with its least-squares interpretation.

The expectations kernel is

$$\boldsymbol{e}=A^{\top+}A^{\top}\mathbf{1}=\left[\left(AA^{\top}\right)^{-1}A\right]A^{\top}\mathbf{1}=\mathbf{1},$$

as required.

Standard Consumer Theory

Via the pricing kernel one can effectively reduce portfolio choice to the standard theory of the consumer. The pricing kernel plays the role of the price vector.

For an investor who wants to maximize utility u, in which utility is defined on the state space, his portfolio choice solves the following problem.

Problem 6 (Portfolio Choice)

$$\max_{\boldsymbol{x}} u\left(A\boldsymbol{x}\right)$$

subject to the budget constraint

$$\langle \boldsymbol{v},\boldsymbol{x}\rangle = w.$$

This problem is almost the standard theory of the consumer, but not quite, since the payoff transformation A is present.

Clearly a utility function u defined on the state space induces a utility function u defined only on the payoff space.

Differential for Utility

If u varies smoothly as the payoff changes, one can define a differential $u_{\boldsymbol{y}}$ defined in the payoff space that shows how utility varies as the payoff changes: if the payoff changes by a small amount $\boldsymbol{y}\Delta t$ (here Δt is very small), then to first order utility changes by

$$\langle u_{\boldsymbol{y}}, \boldsymbol{y} \rangle \, \Delta t.$$

Even though there is no natural basis in the payoff space, a payoff $\boldsymbol{y}$ and the differential $u_{\boldsymbol{y}}$ are nevertheless uniquely defined, coordinate-free vectors.

Standard Consumer Choice

One can redefine the portfolio choice problem (6) as standard consumer choice.

Problem 7 (Portfolio Choice)

$$\max_{\boldsymbol{y}} u(\boldsymbol{y})$$

subject to the budget constraint

$$\langle \boldsymbol{p}, \boldsymbol{y} \rangle = w.$$

The first-order condition for utility maximization is the standard condition

$$u_{\boldsymbol{y}} \propto \boldsymbol{p},$$

in which the positive proportionality factor is the marginal utility of wealth.