Financial Economics

Pricing Kernel and Expectations Kernel

**Pricing Kernel**

By the law of one price, two portfolios with the same payoff $y$ have the same cost. Cost is linear, so there exists some linear function $\langle p, y \rangle$ on the payoff space that equals the cost of the portfolio. Necessarily $p \in \mathbb{R}(A)$ is unique.

For a portfolio $x$ such that $y = Ax$, the cost

$$\langle v, x \rangle = \langle p, y \rangle = \langle p, Ax \rangle,$$

so $p$ is a stochastic discount factor. One refers to $p$ as the **pricing kernel**.

**Definition 1 (Pricing Kernel)** *The pricing kernel is the unique stochastic discount factor in the payoff space.*

**Theorem 2 (Pricing Kernel)** *The pricing kernel is*

$$p = A^\top + v. \quad (2)$$

Even if there is no opportunity for profitable arbitrage, it is not necessary that $p \gg 0$.

**Least-Squares Interpretation**

Consider the least-squares linear regression of the dependent variable $v$ on the independent variables $A^\top$ with regression coefficients $p$.

**Problem 3 (Pricing Kernel as Least Squares)**

$$\min_p \langle v - A^\top p, v - A^\top p \rangle.$$  

The solution set for the regression coefficients is (1), and (2) is the unique solution in $\mathbb{R}(A)$.  

**Formula for the Pricing Kernel**

For any stochastic discount factor,

$$A^\top y = v.$$

There exists a solution $y$ if and only if the law of one price holds. The solution set is then

$$\{ A^\top + v \} + \text{N}(A^\top). \quad (1)$$

By the fundamental theorem of linear algebra, $\text{N}(A^\top)$ and the payoff space $\mathbb{R}(A)$ are orthogonal, so the pricing kernel must be the expression in braces.
Expectations Kernel

The expected payoff on portfolio $x$ is $\langle 1, Ax \rangle$.

Since expectation is linear, the expected payoff is some linear function $\langle e, y \rangle$ on the payoff space, for a unique $e$.

One refers to $e$ as the *expectations kernel*.

For any $x$, $e$ must satisfy

$$\langle 1, Ax \rangle = \langle e, Ax \rangle,$$

so

$$\langle A^\top 1, x \rangle = \langle A^\top e, x \rangle.$$

Since this condition must hold for any $x$, therefore

$$A^\top e = A^\top 1. \quad (3)$$

Since $e$ necessarily exists, this equation has a solution. The solution set is

$$\left\{ A^\top + A^\top 1 \right\} + N \left( A^\top \right).$$

By the fundamental theorem of linear algebra, $N \left( A^\top \right)$ is orthogonal to the payoff space $R \left( A \right)$, so $e$ must be the expression in braces.

Theorem 4 (Expectations Kernel)

$$e = A^\top + A^\top 1.$$

Least-Squares Interpretation

One can interpret the expectations kernel as the solution to a least-squares problem.

Problem 5 (Expectations Kernel as Least Squares)

$$\min_{e \in R \left( A \right)} \langle 1 - e, 1 - e \rangle.$$

Interpret $e$ as the fitted values of the least-squares linear regression of the dependent variable $1$ on the dependent variables $A$. That $e \in R \left( A \right)$ means that $e = Ax$ for regression coefficients $x$. 
The condition (3) says that the residual $1 - e$ is orthogonal to the dependent variables. Hence the necessary and sufficient conditions for least-squares linear regression are fulfilled. The expectations kernel (3) is the unique solution to least-squares problem (5).

\[ \langle e, e \rangle \leq 1, \]
with equality only if $e = 1$.

The pricing kernel is then
\[ p = A^\top v = \left( AA^\top \right)^{-1} A v, \]
in accord with its least-squares interpretation.

The expectations kernel is
\[ e = \left( AA^\top \right)^{-1} A \]
as required.

For an investor who wants to maximize utility $u$, in which utility is defined on the state space, his portfolio choice solves the following problem.

**Problem 6 (Portfolio Choice)**

\[ \max_x u \left( A x \right) \]

subject to the budget constraint
\[ \langle v, x \rangle = w. \]

This problem is almost the standard theory of the consumer, but not quite, since the payoff transformation $A$ is present.
Clearly a utility function \( u \) defined on the state space induces a utility function \( u \) defined only on the payoff space.

### Differential for Utility

If \( u \) varies smoothly as the payoff changes, one can define a differential \( u_y \) defined in the payoff space that shows how utility varies as the payoff changes: if the payoff changes by a small amount \( y \Delta t \) (here \( \Delta t \) is very small), then to first order utility changes by

\[
\langle u_y, y \rangle \Delta t.
\]

Even though there is no natural basis in the payoff space, a payoff \( y \) and the differential \( u_y \) are nevertheless uniquely defined, coordinate-free vectors.

### Standard Consumer Choice

One can redefine the portfolio choice problem (6) as standard consumer choice.

**Problem 7 (Portfolio Choice)**

\[
\max_y u(y)
\]

subject to the budget constraint

\[
\langle p, y \rangle = w.
\]