Black-Scholes Option Pricing

The pricing kernel furnishes an alternate derivation of the Black-Scholes formula for the price of a call option. Arbitrage is again the foundation for the theory.

Risk-Free Asset and Stock

The risk-free asset has the return

$$r\,\mathrm{d}t;$$

a one-dollar investment at t is worth $1 + r\,\mathrm{d}t$ at $t + \mathrm{d}t$.

The stock has the return

$$\frac{\mathrm{d}s_t}{s_t} = (r + \mu)\,\mathrm{d}t + \sigma\,\mathrm{d}z_t;$$

a one-dollar investment at t is worth $1 + (r + \mu)\,\mathrm{d}t + \sigma\,\mathrm{d}z_t$ at $t + \mathrm{d}t$.

Stochastic Discount Factor

Let p_t denote a stochastic discount factor.

For an asset with price q_t and future payments d_t,

$$p_t q_t = \int_t^\infty \mathrm{E}_t \left(p_\tau d_\tau \right) \mathrm{d}\tau,$$

the present discounted value of the future payments.

Then

$$\begin{aligned} p_t q_t &= p_t d_t \,\mathrm{d}t + \int_{t+\mathrm{d}t}^{\infty} \mathrm{E}_t \left(p_\tau d_\tau \right) \mathrm{d}\tau \\ &= p_t d_t \,\mathrm{d}t + \mathrm{E}_t \left(\int_{t+\mathrm{d}t}^{\infty} p_\tau d_\tau \,\mathrm{d}\tau \right) \\ &= p_t d_t \,\mathrm{d}t + \mathrm{E}_t \left(p_{t+\mathrm{d}t} q_{t+\mathrm{d}t} \right). \end{aligned}$$

Stock Pricing

For the stochastic discount factor to price the stock,

$$p_t s_t = \mathrm{E}_t \left(p_{t+\mathrm{d}t} s_{t+\mathrm{d}t} \right).$$

Hence

$$\begin{aligned} p_t s_t &= \mathrm{E}_t \left(p_{t+\mathrm{d}t} s_{t+\mathrm{d}t} \right) \\ &= \mathrm{E}_t \left[\left(p_t + \mathrm{d}p_t \right) \left(s_t + \mathrm{d}s_t \right) \right] \\ &= \mathrm{E}_t \left(p_t s_t + s_t \, \mathrm{d}p_t + p_t \, \mathrm{d}s_t + \mathrm{d}p_t \, \mathrm{d}s_t \right). \end{aligned}$$

Dividing by $p_t s_t$ and cancelling gives

$$0 = \mathrm{E}_t \left(\frac{\mathrm{d}p_t}{p_t} + \frac{\mathrm{d}s_t}{s_t} + \frac{\mathrm{d}p_t}{p_t} \frac{\mathrm{d}s_t}{s_t} \right). \tag{1}$$

Risk-Free Asset Pricing

For the stochastic discount factor to price the risk-free asset,

$$p_t = \mathrm{E}_t \left[p_{t+\mathrm{d}t} \left(1 + r\,\mathrm{d}t \right) \right],$$

so

$$\begin{aligned} p_t &= \mathrm{E}_t \left[\left(p_t + \mathrm{d}p_t \right) \left(1 + r\,\mathrm{d}t \right) \right] \\ &= \mathrm{E}_t \left(p_t + p_t r\,\mathrm{d}t + \mathrm{d}p_t \right), \end{aligned}$$

since the second-order term is zero. Dividing by p_t and cancelling gives

$$0 = \mathrm{E}_t \left(r\,\mathrm{d}t + \frac{\mathrm{d}p_t}{p_t} \right). \tag{2}$$

Pricing Kernel

The pricing kernel p_t is a stochastic discount factor of the form

$$\frac{\mathrm{d}p_t}{p_t} = a\,\mathrm{d}t + b\,\mathrm{d}z_t\,,$$

the span of the returns on the risk-free asset and the stock.

By (2), for the pricing kernel to price the risk-free asset requires $a = -r$.

By (1), for the pricing kernel to price the stock requires

$$
\begin{aligned}
0 &= \mathrm{E}_t\left(\frac{\mathrm{d}p_t}{p_t} + \frac{\mathrm{d}s_t}{s_t} + \frac{\mathrm{d}p_t}{p_t}\frac{\mathrm{d}s_t}{s_t}\right) \\
&= \mathrm{E}_t\left\{(-r\,\mathrm{d}t + b\,\mathrm{d}z_t) + [(r+\mu)\,\mathrm{d}t + \sigma\,\mathrm{d}z_t]\right. \\
&\quad \left. + (-r\,\mathrm{d}t + b\,\mathrm{d}z_t)[(r+\mu)\,\mathrm{d}t + \sigma\,\mathrm{d}z_t]\right\} \\
&= \mathrm{E}_t\left[-r\,\mathrm{d}t + (r+\mu)\,\mathrm{d}t + b\sigma\,\mathrm{d}t\right],
\end{aligned}
$$

so $b = -\mu/\sigma$.

Thus the pricing kernel follows the stochastic differential equation

$$\frac{\mathrm{d}p_t}{p_t} = -r\,\mathrm{d}t - \frac{\mu}{\sigma}\mathrm{d}z_t\,.$$

For the initial condition $p_0 = 1$, the solution is

$$\ln p_t = \left[-r - \frac{1}{2}\left(\frac{\mu}{\sigma}\right)^2\right]t - \frac{\mu}{\sigma}z_t\,.$$

Arbitrage

Following Black and Scholes, assume that the call price c_t is a function of the stock price. Then its return lies in the span of the returns of the stock price and the risk-free asset.

The absence of arbitrage then requires that the return on the call can be priced by the pricing kernel for the stock and the risk-free asset.

Black-Scholes Partial Differential Equation

If $c_t = c(s_t, \tau)$ (τ is the time to expiration), by the second-order Taylor series expansion

$$\begin{aligned} \mathrm{d}c_t &= -c_\tau \, \mathrm{d}t + c_s \, \mathrm{d}s_t + \frac{1}{2} c_{ss} (\mathrm{d}s_t)^2 \\ &= -c_\tau \, \mathrm{d}t + c_s s_t \left[(r+\mu) \, \mathrm{d}t + \sigma \, \mathrm{d}z_t \right] \\ &+ \frac{1}{2} c_{ss} s_t^2 \left[(r+\mu) \, \mathrm{d}t + \sigma \, \mathrm{d}z_t \right]^2 . \end{aligned}$$

For the pricing kernel to price the call,

$$0 = \mathrm{E}_t \left(\frac{\mathrm{d}p_t}{p_t} + \frac{\mathrm{d}c_t}{c_t} + \frac{\mathrm{d}p_t}{p_t} \frac{\mathrm{d}c_t}{c_t} \right).$$

Hence

$$0 = \mathrm{E}_t \left[\left(-r\,\mathrm{d}t - \frac{\mu}{\sigma} \mathrm{d}z_t \right) + \frac{\mathrm{d}c_t}{c_t} + \left(-r\,\mathrm{d}t - \frac{\mu}{\sigma} \mathrm{d}z_t \right) \frac{\mathrm{d}c_t}{c_t} \right].$$

$$= \left\{ -r + \frac{1}{c_t} \left[-c_\tau + c_s s_t \left(r + \mu \right) + \frac{1}{2} c_{ss} s_t^2 \sigma^2 - \frac{\mu}{\sigma} c_s s_t \sigma \right] \right\} \mathrm{d}t,$$

which yields the Black-Scholes partial differential equation

$$0 = -r\,c_t - c_\tau + c_s s_t\,r + \frac{1}{2} c_{ss} s_t^2 \sigma^2.$$

(Here c_t is the call price at time t, but c_τ is the partial derivative of the price with respect to τ.)

Present Discounted Value

Equivalently, the call price is the present value of its exercise value at expiration, using the pricing kernel as the stochastic discount factor.

Theorem 1

$$c(s_0, t) = \mathrm{E}_0\left[p_t c(s_t, 0)\right]. \tag{3}$$

Here

$$c(s_t, 0) = \max\left[s_t - x, 0\right],$$

the value at expiration with striking price x.

Computation of the Expected Value

Since

$$s_t = s_0 \exp\left[\left(r + \mu - \frac{1}{2}\sigma^2\right)t + \sigma z_t\right],$$

therefore $s_t \geq x$ for

$$z_t \geq \frac{1}{\sigma}\left[\ln(x/s_0) - \left(r + \mu - \frac{1}{2}\sigma^2\right)t\right] := \underline{z}. \tag{4}$$

From our previous work,

$$p_t s_t = s_0 \exp\left[-\frac{1}{2}\left(\sigma - \frac{\mu}{\sigma}\right)^2 t + \left(\sigma - \frac{\mu}{\sigma}\right) z_t\right] \tag{5}$$

$$p_t x = x\mathrm{e}^{-rt} \exp\left[-\frac{1}{2}\left(\frac{\mu}{\sigma}\right)^2 t - \frac{\mu}{\sigma} z_t\right]. \tag{6}$$

We calculate the expected value of these expressions over the range (4).

The probability density function of z_t is

$$\frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2} z_t^2 / t\right).$$

When one integrates to find the expectations, the quadratic in z_t combines with the terms linear in z_t in the exponentials (5)-(6) to form a quadratic. This quadratic is again a normal probability density function, still with variance t, but the mean is non-zero.

$$
\begin{aligned}
&\mathrm{E}_0\left(p_t s_t\right) \text{ over } z_t \geq \underline{z} \\
&= \int_{\underline{z}}^{\infty} \left\{ s_0 \exp\left[-\frac{1}{2}\left(\sigma - \frac{\mu}{\sigma}\right)^2 t + \left(\sigma - \frac{\mu}{\sigma}\right) z \right] \right. \\
&\qquad \left. \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2} z^2 / t \right) \mathrm{d}z \right\} \\
&= s_0 \int_{\underline{z}}^{\infty} \frac{1}{\sqrt{2\pi t}} \exp\left\{ -\frac{1}{2}\left[z - \left(\sigma - \frac{\mu}{\sigma}\right) t \right]^2 / t \right\} \mathrm{d}z \\
&= s_0 F\left\{ \left[\left(\sigma - \frac{\mu}{\sigma}\right) t - \underline{z} \right] / \sqrt{t} \right\}
\end{aligned}
$$

in which F is the cumulative distribution function for a normal with mean zero and variance one.

Substituting for $\underline{z}$ gives

$$
\begin{aligned}
&\mathrm{E}_0\left(p_t s_t\right) \text{ over } z_t \geq \underline{z} \\
&\quad = s_0 F\left\{\left(\sigma - \frac{\mu}{\sigma}\right)\sqrt{t} \right. \\
&\qquad \left. + \left[\ln\left(s_0/x\right) + \left(r + \mu - \frac{1}{2}\sigma^2\right)t\right]/\sigma\sqrt{t}\right\} \\
&\quad = s_0 F\left\{\left[\ln\left(s_0/x\right) + \left(r + \frac{1}{2}\sigma^2\right)t\right]/\sigma\sqrt{t}\right\}.
\end{aligned}
$$

Here μ has cancelled out!

$$
\begin{aligned}
&\mathrm{E}_0\left(p_t x\right) \text{ over } z_t \geq \underline{z} \\
&= \int_{\underline{z}}^{\infty} \left\{ x\mathrm{e}^{-rt} \exp\left[-\frac{1}{2}\left(\frac{\mu}{\sigma}\right)^2 t - \frac{\mu}{\sigma} z\right] \right. \\
&\quad \left. \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2} z^2 / t\right) \mathrm{dz} \right\} \\
&= x\mathrm{e}^{-rt} \int_{\underline{z}}^{\infty} \frac{1}{\sqrt{2\pi t}} \exp\left[-\frac{1}{2}\left(z + \frac{\mu}{\sigma} t\right)^2 / t\right] \mathrm{dz} \\
&= x\mathrm{e}^{-rt} F\left[\left(-\frac{\mu}{\sigma} t - \underline{z}\right) / \sqrt{t}\right].
\end{aligned}
$$

Substituting for $\underline{z}$ gives

$$\begin{aligned} \mathrm{E}_0\left(p_t x\right) \text{ over } z_t \geq \underline{z} & \\ &= x\mathrm{e}^{-rt} F\left\{-\frac{\mu}{\sigma}\sqrt{t} + \left[\ln\left(s_0/x\right) + \left(r + \mu - \frac{1}{2}\sigma^2\right)t\right]/\sigma\sqrt{t}\right\} \\ &= x\mathrm{e}^{-rt} F\left\{\left[\ln\left(s_0/x\right) + \left(r - \frac{1}{2}\sigma^2\right)t\right]/\sigma\sqrt{t}\right\}. \end{aligned}$$

Again μ has cancelled out!

Black-Scholes Formula

The price of the call option is the difference in the two present discounted values.

Theorem 2 (Black-Scholes) *The price of the call option is*

$$
\begin{aligned}
&\mathrm{E}_0\left(p_t \max\left[s_t - x, 0\right]\right) \\
&\quad = s_0 F\left\{\left[\ln\left(s_0/x\right) + \left(r + \frac{1}{2}\sigma^2\right)t\right]/\sigma\sqrt{t}\right\} \\
&\qquad - x\mathrm{e}^{-rt} F\left\{\left[\ln\left(s_0/x\right) + \left(r - \frac{1}{2}\sigma^2\right)t\right]/\sigma\sqrt{t}\right\}.
\end{aligned}
$$