Budget Constraint

Suppose that all consumption is financed out of wealth; there is no labor income or other income. The budget constraint is

$$\begin{aligned} w_{t+\mathrm{d}t} &= (w_t - c_t\,\mathrm{d}t)\,(1+\mathrm{d}a_t) \\ &= w_t\,(1+\mathrm{d}a_t) - c_t\,\mathrm{d}t. \end{aligned}$$

Equivalently,

$$\mathrm{d}w_t = w_{t+\mathrm{d}t} - w_t = w_t\,\mathrm{d}a_t - c_t\,\mathrm{d}t. \tag{1}$$

Asset Return

Assume that the probability distribution of the return $\mathrm{d}a_t$ is independent of time,

$$\mathrm{d}a_t = m\,\mathrm{d}t + s\,\mathrm{d}z_t. \tag{2}$$

There is no portfolio choice; the consumer invests wealth with this return.

Constant Relative Risk Aversion

The consumer maximizes expected utility

$$\mathrm{E}_t\left[\int_t^\infty v\,(c_\tau)\,\mathrm{e}^{-\rho(\tau-t)}\,\mathrm{d}\tau\right] = \mathrm{E}_t\left[\int_t^\infty \frac{c^{1-\alpha}}{1-\alpha}\mathrm{e}^{-\rho(\tau-t)}\,\mathrm{d}\tau\right].$$

There is constant relative risk aversion, $\alpha > 0, \alpha \neq 1$.

Consumption Decision

By the proportionality property of constant relative risk aversion, it is intuitive that the optimum consumption decision is to consume a fixed fraction of wealth,

$$c_t = k w_t. \tag{3}$$

Here k is constant, and we determine the optimum value. High k means low saving.

Budget Constraint

Substitute the feedback rule (3) into the budget constraint (1):

$$\mathrm{d}\,(c_t/k) = (c_t/k)\,\mathrm{d}a_t - c_t\,\mathrm{d}t,$$

so

$$\frac{\mathrm{d}c_t}{c_t} = \mathrm{d}a_t - k\,\mathrm{d}t. \tag{4}$$

From our earlier work, a necessary condition for optimum consumption is

$$\begin{aligned} 0 = \mathrm{E}_t\,(\mathrm{d}a_t - \rho\,\mathrm{d}t) + \mathrm{E}_t\left[-\alpha\left(\frac{\mathrm{d}c_t}{c_t}\right)(1+\mathrm{d}a_t)\right] \\ +\,\mathrm{E}_t\left[\frac{1}{2}\alpha\,(\alpha+1)\left(\frac{\mathrm{d}c_t}{c_t}\right)^2\right]. \end{aligned}$$

Substituting (4) gives

$$\begin{aligned}
0 &= \mathrm{E}_t\left(\mathrm{d}a_t - \rho\,\mathrm{d}t\right) + \mathrm{E}_t\left[-\alpha\left(\mathrm{d}a_t - k\,\mathrm{d}t\right)\left(1 + \mathrm{d}a_t\right)\right] \\
&\quad + \mathrm{E}_t\left[\frac{1}{2}\alpha\left(\alpha+1\right)\left(\mathrm{d}a_t - k\,\mathrm{d}t\right)^2\right] \text{ by (4)} \\
&= \mathrm{E}_t\left(\mathrm{d}a_t\right) - \rho\,\mathrm{d}t + \mathrm{E}_t\left[-\alpha\,\mathrm{d}a_t\left(1+\mathrm{d}a_t\right)\right] + \alpha k\,\mathrm{d}t \\
&\quad + \frac{1}{2}\alpha\left(\alpha+1\right)\left(\mathrm{d}a_t\right)^2 \\
&= \left(1-\alpha\right)m\,\mathrm{d}t + \left(\alpha k - \rho\right)\mathrm{d}t + \left[-\alpha + \frac{1}{2}\alpha\left(\alpha+1\right)\right]s^2\,\mathrm{d}t \text{ by (2).}
\end{aligned}$$

Cancelling gives $\mathrm{d}t$ and solving for k gives

$$k = \frac{1}{\alpha}\left[\rho + \left(\alpha-1\right)m - \frac{1}{2}\alpha\left(\alpha-1\right)s^2\right]. \qquad (5)$$

Then

$$\begin{aligned}
\frac{\mathrm{d}c_t}{c_t} &= \mathrm{d}a_t - k\,\mathrm{d}t \\
&= \left(m\,\mathrm{d}t + s\,\mathrm{d}z_t\right) - \left\{\frac{1}{\alpha}\left[\rho + \left(\alpha-1\right)m - \frac{1}{2}\alpha\left(\alpha-1\right)s^2\right]\right\}\mathrm{d}t \\
&= \frac{1}{\alpha}\left[\left(m-\rho\right) + \frac{1}{2}\alpha\left(\alpha-1\right)s^2\right]\mathrm{d}t + s\,\mathrm{d}z_t.
\end{aligned}$$

Certainty

For $s = 0$, we have the result with certainty,

$$\begin{aligned}
k &= \frac{1}{\alpha}\left[\rho + \left(\alpha-1\right)m\right] \\
\frac{\mathrm{d}c_t}{c_t} &= \frac{1}{\alpha}\left(m-\rho\right)\mathrm{d}t.
\end{aligned}$$

Substitution and Wealth Effects

Above the expression

$$\alpha - 1$$

reflects the opposition of substitution and wealth effects. A small value α means high intertemporal substitution. Thus $\alpha - 1 < 0$ means that the substitution effect dominates.

For example, in (5), if the substitution effect dominates, then higher mean return m raises saving and greater uncertainty s reduces saving (but increases the mean growth rate of consumption).

Logarithmic Utility

These formulas also apply for logarithmic utility, $\alpha = 1$,

$$v\left(c_t\right) = \ln c_t.$$

The substitution and wealth effects cancel, and

$$\begin{aligned}
k &= \rho \\
c_t &= \rho w_t \\
\frac{\mathrm{d}c_t}{c_t} &= \left(m-\rho\right)\mathrm{d}t + s\,\mathrm{d}z_t.
\end{aligned}$$