Budget Constraint

Suppose that all consumption is financed out of wealth; there is no labor income or other income. The budget constraint is

\[ w_{t+dt} = (w_t - c_t \, dt)(1 + da_t) \]
\[ = w_t (1 + da_t) - c_t \, dt. \]

Equivalently,

\[ dw_t = w_{t+dt} - w_t = w_t \, da_t - c_t \, dt. \]  \hspace{1cm} (1)
Asset Return

Assume that the probability distribution of the return $d a_t$ is independent of time,

$$d a_t = m d t + s d z_t.$$  \hspace{1cm} (2)

There is no portfolio choice; the consumer invests wealth with this return.
The consumer maximizes expected utility

\[ E_t \left[ \int_t^\infty v(c_\tau) e^{-\rho(\tau-t)} d\tau \right] = E_t \left[ \int_t^\infty \frac{c^{1-\alpha}}{1-\alpha} e^{-\rho(\tau-t)} d\tau \right]. \]

There is constant relative risk aversion, \( \alpha > 0, \alpha \neq 1. \)
Consumption Decision

By the proportionality property of constant relative risk aversion, it is intuitive that the optimum consumption decision is to consume a fixed fraction of wealth,

\[ c_t = kw_t. \]  \hspace{1cm} (3)

Here \( k \) is constant, and we determine the optimum value. High \( k \) means low saving.
Budget Constraint

Substitute the feedback rule (3) into the budget constraint (1):

\[ d \left( \frac{c_t}{k} \right) = \left( \frac{c_t}{k} \right) da_t - c_t \, dt, \]

so

\[ \frac{dc_t}{c_t} = da_t - k \, dt. \quad (4) \]
From our earlier work, a necessary condition for optimum consumption is

\[ 0 = E_t (d\alpha_t - \rho dt) + E_t \left[ -\alpha \left( \frac{dc_t}{c_t} \right) (1 + d\alpha_t) \right] \]

\[ + E_t \left[ \frac{1}{2} \alpha (\alpha + 1) \left( \frac{dc_t}{c_t} \right)^2 \right]. \]
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Substituting (4) gives

\[
0 = E_t (da_t - \rho \, dt) + E_t \left[ -\alpha (da_t - k \, dt) (1 + da_t) \right] \\
+ E_t \left[ \frac{1}{2} \alpha (\alpha + 1) (da_t - k \, dt)^2 \right] \quad \text{by (4)} \\
= E_t (da_t) - \rho \, dt + E_t \left[ -\alpha da_t (1 + da_t) \right] + \alpha k \, dt \\
+ \frac{1}{2} \alpha (\alpha + 1) (da_t)^2 \\
= (1 - \alpha) m \, dt + (\alpha k - \rho) \, dt + \left[ -\alpha + \frac{1}{2} \alpha (\alpha + 1) \right] s^2 \, dt \quad \text{by (2)}
\]
Cancelling gives \( dt \) and solving for \( k \) gives

\[
k = \frac{1}{\alpha} \left[ \rho + (\alpha - 1) m - \frac{1}{2} \alpha (\alpha - 1) s^2 \right]. \tag{5}
\]

Then

\[
\frac{dc_t}{c_t} = da_t - k \, dt \\
= (mdt + s \, dz_t) - \left\{ \frac{1}{\alpha} \left[ \rho + (\alpha - 1) m - \frac{1}{2} \alpha (\alpha - 1) s^2 \right] \right\} \, dt \\
= \frac{1}{\alpha} \left[ (m - \rho) + \frac{1}{2} \alpha (\alpha - 1) s^2 \right] \, dt + s \, dz_t.
\]
Certainty

For $s = 0$, we have the result with certainty,

$$k = \frac{1}{\alpha} \left[ \rho + (\alpha - 1)m \right]$$

$$\frac{dc_t}{c_t} = \frac{1}{\alpha} (m - \rho) \, dt.$$
Substitution and Wealth Effects

Above the expression

$$\alpha - 1$$

reflects the opposition of substitution and wealth effects. A small value $\alpha$ means high intertemporal substitution. Thus $\alpha - 1 < 0$ means that the substitution effect dominates.

For example, in (5), if the substitution effect dominates, then higher mean return $m$ raises saving and greater uncertainty $s$ reduces saving (but increases the mean growth rate of consumption).
Logarithmic Utility

These formulas also apply for logarithmic utility, $\alpha = 1$,

$$v(c_t) = \ln c_t.$$ 

The substitution and wealth effects cancel, and

$$k = \rho$$

$$c_t = \rho w_t$$

$$\frac{dc_t}{c_t} = (m - \rho) \, dt + s \, dz_t.$$