Tradeoff of Mean and Variance

Mean-Variance: simple model of choice under uncertainty.

More is better: a higher mean is desirable.

Risk aversion makes a higher variance undesirable.

A higher variance is offset by a higher mean.

The tradeoff depends on the degree of risk aversion.

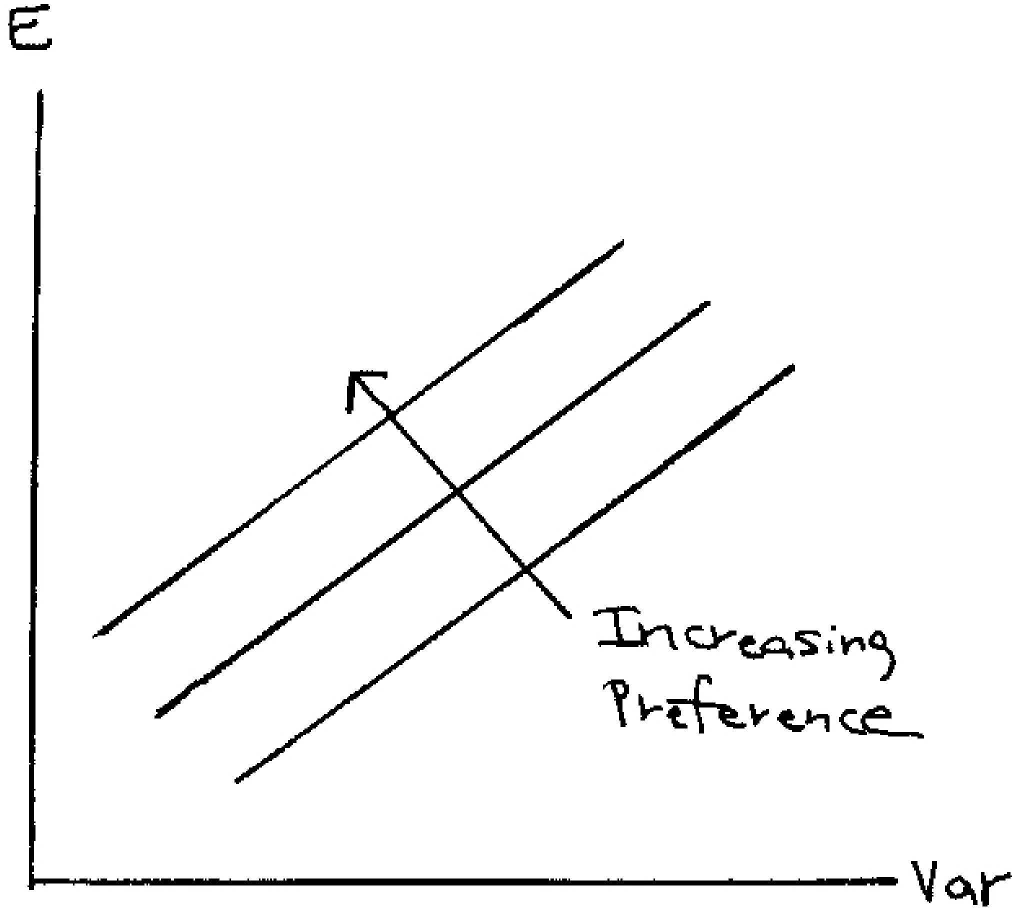

Figure 1: Indifference Curves for Mean-Variance

Inconsistency in Mean-Variance

Borch [1] shows that unrestricted mean-variance contradicts the fundamental principle that more is better. Suppose that

$$(E, S) \text{ and } (E + \Delta E, S + \Delta S)$$

lie on the same mean/standard deviation indifference curve. For simplicity, assume $\Delta E = \Delta S > 0$ (Borch treats the general case).

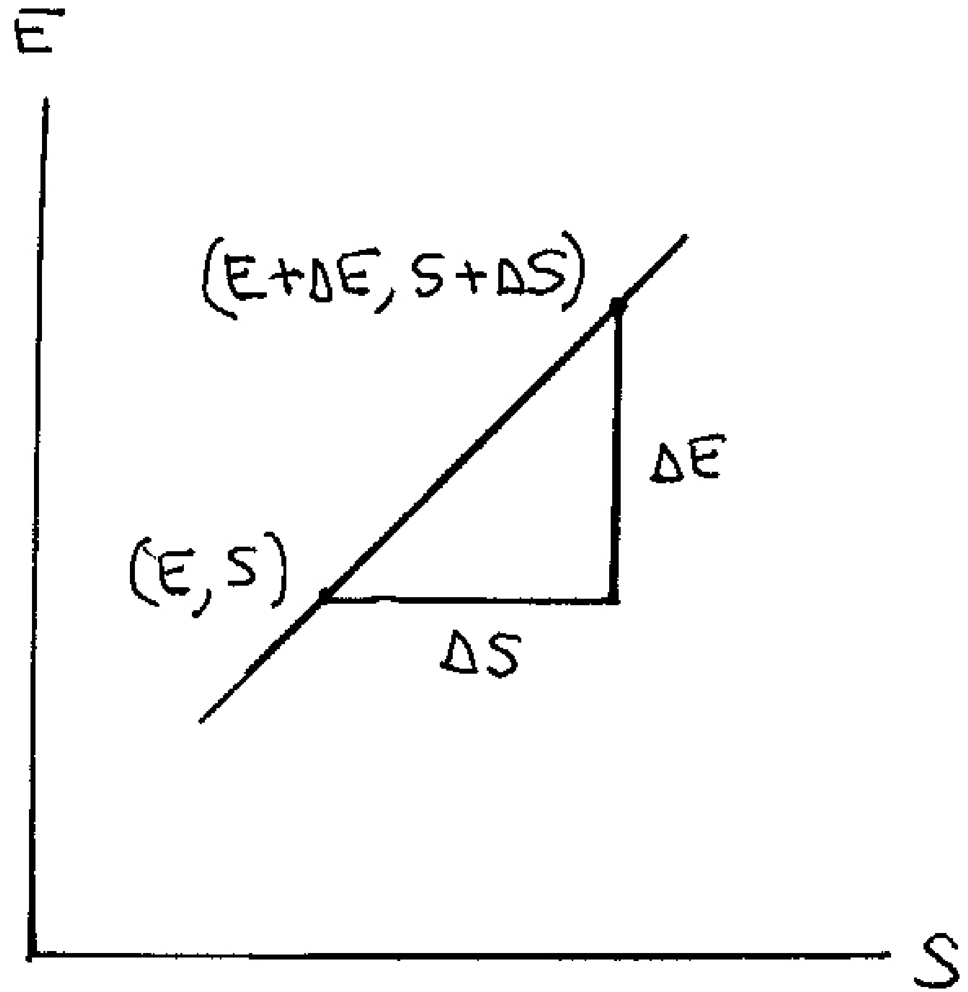

Figure 2: Indifference Curve for Mean-Standard Deviation

Compare the following two gambles.

Gamble (E, S):
$E - S$ with probability $\frac{1}{2}$;
$E + S$ with probability $\frac{1}{2}$.

Gamble $(E + \Delta E, S + \Delta S)$:
$E - S$ with probability $\frac{1}{2}$;
$E + S + 2\Delta E$ with probability $\frac{1}{2}$.

Clearly the second dominates the first, contradicting the supposition that both lie on the same indifference curve.

Justification for Mean-Variance

- Normally distributed outcomes;
- Quadratic utility;
- Small risks.

Normally Distributed Outcomes

For normally distributed outcomes, the mean and the variance completely describe the probability distribution. Regardless of the utility function, the expected utility is a function of the mean and the variance only.

Quadratic Utility

If utility is quadratic, then expected utility is determined by the mean and the variance only, regardless of the probability distribution of the outcomes.

By taking an appropriate linear transformation, any quadratic utility can be reduced to the form

$$u(w) = -(w - \widetilde{w})^2 .$$

The pertinent region is $w \leq \widetilde{w}$, as marginal utility is negative at higher values.

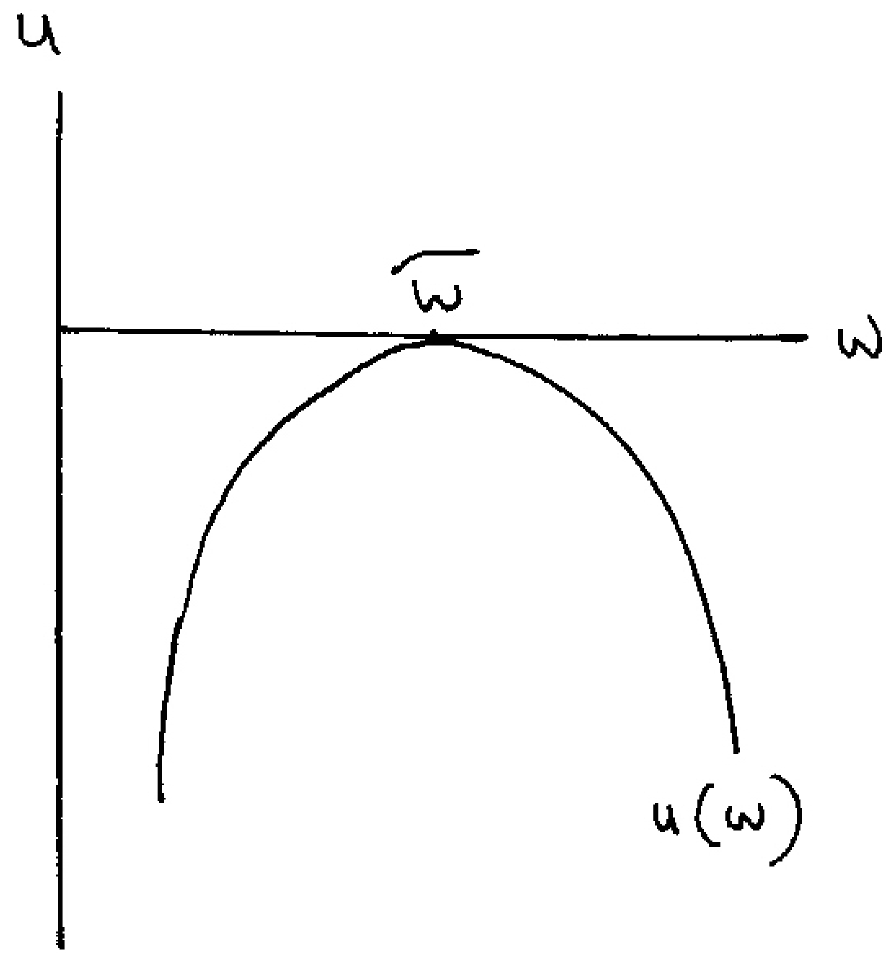

Figure 3: Quadratic Utility

Expected utility is

$$\begin{aligned}
\mathrm{E}\left[u\left(w\right)\right] &= \mathrm{E}\left[-\left(w-\widetilde{w}\right)^2\right] \\
&= \mathrm{E}\left\{-\left[w-\mathrm{E}\left(w\right)+\mathrm{E}\left(w\right)-\widetilde{w}\right]^2\right\} \\
&= \mathrm{E}\left\{-\left[w-\mathrm{E}\left(w\right)\right]^2-2\left[w-\mathrm{E}\left(w\right)\right]\left[\mathrm{E}\left(w\right)-\widetilde{w}\right]\right. \\
&\quad \left.-\left[\mathrm{E}\left(w\right)-\widetilde{w}\right]^2\right\} \\
&= -\left\{\mathrm{Var}\left(w\right)+\left[\mathrm{E}\left(w\right)-\widetilde{w}\right]^2\right\},
\end{aligned}$$

a function only of the mean and the variance.

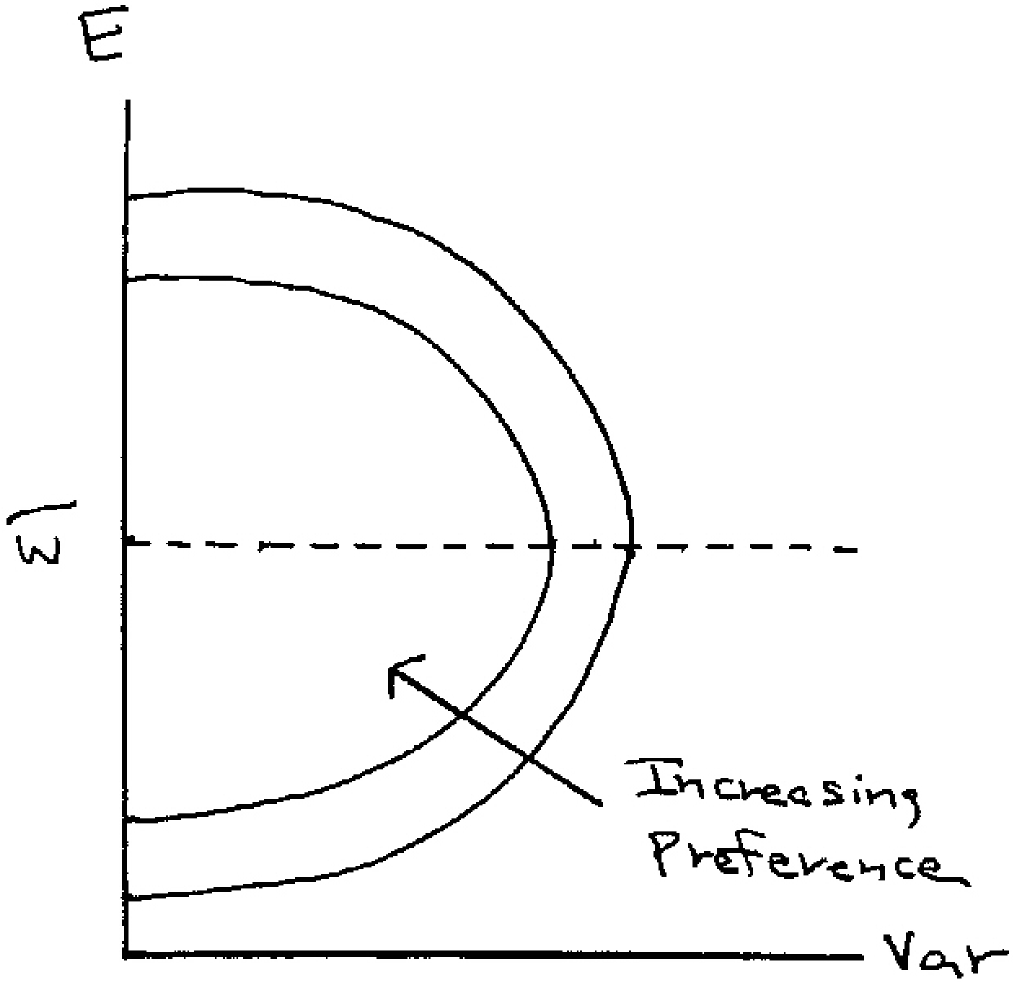

Figure 4: Indifference Curves for Quadratic Utility

Small Risks

For a short time horizon, the change in the value of a portfolio is small. One can approximate utility by its second-order Taylor series,

$$u(w) \approx u(\widetilde{w}) + u'(\widetilde{w})(w - \widetilde{w}) + \frac{1}{2}u''(\widetilde{w})(w - \widetilde{w})^2.$$

Utility is approximately quadratic, so expected utility is determined by the mean and the variance.

References

[1] Karl Borch. A note on uncertainty and indifference curves. *Review of Economic Studies*, XXXVI(1)(105):1–4, January 1969. HB1R4.