Calculus Rules

In standard, non-stochastic calculus, one computes a derivative or an integral using various rules. In the Itô stochastic calculus, one extends these rules to the stochastic terms.

Suppose that u is some function $u(x)$ of x. We want to express the differential $\mathrm{d}u$ in terms of the differential $\mathrm{d}x$.

Taylor Series

Consider the Taylor series expansion of $u(x)$ about some value $\bar{x}$:

$$u(x) = u(\bar{x}) + u'(\bar{x})(x - \bar{x}) + \frac{1}{2}u''(\bar{x})(x - \bar{x})^2 + \frac{1}{3!}u'''(\bar{x})(x - \bar{x})^3 + \cdots$$

Under certain general conditions, $u(x)$ equals this infinite sum exactly.

Rewrite this expression in terms of the changes $\Delta x := x - \bar{x}$ and $\Delta u := u(x) - u(\bar{x})$:

$$\Delta u = u'(\bar{x})\Delta x + \frac{1}{2}u''(\bar{x})(\Delta x)^2 + \frac{1}{3!}u'''(\bar{x})(\Delta x)^3 + \cdots .$$

Replace the difference by the differential:

$$\mathrm{d}u = u'(\bar{x})\mathrm{d}x + \frac{1}{2}u''(\bar{x})(\mathrm{d}x)^2 + \frac{1}{3!}u'''(\bar{x})(\mathrm{d}x)^3 + \cdots . \tag{1}$$

Non-Stochastic Calculus

In standard, non-stochastic calculus, one computes a differential simply by keeping the first-order terms. For small changes in the variable, second-order and higher terms are negligible compared to the first-order terms. Equation (1) becomes

$$\mathrm{d}u = u' \, \mathrm{d}x.$$

The change in u is proportional to the change in x.

Stochastic Calculus—Itô’s Formula

In stochastic calculus, one must also keep the second-order terms. Equation (1) becomes *Itô’s formula*,

$$\mathrm{d}u = u' \,\mathrm{d}x + \frac{1}{2} u'' (\mathrm{d}x)^2 \tag{2}$$

This equation is *exact*; the third-order and higher order terms are zero.

Rules of Stochastic Calculus

One computes Itô’s formula (2) using the rules (3). Let z denote Wiener-Brownian motion, and let t denote time. One computes using the rules

$$\begin{aligned} (\mathrm{d}z)^2 &= \mathrm{d}t, \\ \mathrm{d}z\,\mathrm{d}t &= 0, \\ (\mathrm{d}t)^2 &= 0. \end{aligned} \tag{3}$$

The key rule is the first and is what sets stochastic calculus apart from non-stochastic calculus.

Computation

Although we prove the rules (3) below, first let us consider the implication of the rules. One computes mechanically, as in ordinary algebra, but using the rules. The second-order terms cannot be dropped, since $(\mathrm{d}z)^2 = \mathrm{d}t$.

Example

If $\mathrm{d}x = m\,\mathrm{d}t + s\,\mathrm{d}z$, then

$$\begin{aligned}
(\mathrm{d}x)^2 &= (m\,\mathrm{d}t + s\,\mathrm{d}z)^2 \\
&= (m\,\mathrm{d}t)^2 + (s\,\mathrm{d}z)^2 + 2\,(m\,\mathrm{d}t)\,(s\,\mathrm{d}z) \\
&= 0 + s^2\,\mathrm{d}t + 0 \\
&= s^2\,\mathrm{d}t.
\end{aligned}$$

The second-order term is non-zero, as long as the instantaneous stochastic part is non-zero ($s \neq 0$).

Therefore Itô's formula (2) says

$$\begin{aligned} \mathrm{d}u &= u'\,(m\,\mathrm{d}t + s\,\mathrm{d}z) + \frac{1}{2}u''(m\,\mathrm{d}t + s\,\mathrm{d}z)^2 \\ &= u'\,(m\,\mathrm{d}t + s\,\mathrm{d}z) + \frac{1}{2}u''s^2\,\mathrm{d}t \\ &= \left(u'm + \frac{1}{2}u''s^2\right)\mathrm{d}t + u's\,\mathrm{d}z. \end{aligned}$$

Third-Order and Higher-Order Terms

Like non-stochastic calculus, third-order and higher-order terms are zero. For example,

$$(\mathrm{d}x)^3 = \mathrm{d}x\,(\mathrm{d}x)^2 = (m\,\mathrm{d}t + s\,\mathrm{d}z)\,s^2\mathrm{d}t = ms^2\,(\,\mathrm{d}t\,)^2 + s\,\mathrm{d}z\,\mathrm{d}t = 0,$$

applying the rules.

Square of Wiener-Brownian Motion

Consider $u = z^2$:

$$\begin{aligned} \mathrm{d}u &= u'\,\mathrm{d}z + \frac{1}{2}u''\,(\mathrm{d}z)^2 \\ &= 2z\,\mathrm{d}z + \frac{1}{2}2\,(\mathrm{d}z)^2 \\ &= 2z\,\mathrm{d}z + \mathrm{d}t. \end{aligned}$$

Relative to non-stochastic calculus, $\mathrm{d}t$ is an extra term.

Confirmation of Previous Result

Essentially the same calculation confirms our earlier limiting result that $\mathrm{d}u = 2z\,\mathrm{d}z$, with initial value $u(0) = 0$, has the solution $u = z^2 - t$:

$$\begin{aligned}
\mathrm{d}u &= u_z\,\mathrm{d}z + u_t\,\mathrm{d}t + \frac{1}{2}u_{zz}(\mathrm{d}z)^2 + u_{zt}\,\mathrm{d}z\,\mathrm{d}t + \frac{1}{2}u_{tt}(\mathrm{d}t)^2 \\
&= 2z\,\mathrm{d}z + (-1)\,\mathrm{d}t + \frac{1}{2}2(\mathrm{d}z)^2 + 0\,\mathrm{d}z\,\mathrm{d}t + \frac{1}{2}0(\mathrm{d}t)^2 \\
&= 2z\,\mathrm{d}z - \mathrm{d}t + \mathrm{d}t \\
&= 2z\,\mathrm{d}z.
\end{aligned}$$

Stochastic Exponential

If $u = \mathrm{e}^{z-t/2}$, then

$$u_z = u \qquad u_{zz} = u$$

$$u_t = -\frac{1}{2}u \quad u_{zt} = -\frac{1}{2}u \quad u_{tt} = \frac{1}{4}u.$$

The Taylor series is

$$
\begin{aligned}
du &= u_z \, \mathrm{dz} + u_t \, \mathrm{d}t + \frac{1}{2} u_{zz} \, (\mathrm{dz})^2 + u_{zt} \, \mathrm{dz} \, \mathrm{d}t + \frac{1}{2} u_{tt} \, (\mathrm{d}t)^2 \\
&= u \, \mathrm{dz} - \frac{1}{2} u \, \mathrm{d}t + \frac{1}{2} u \, (\mathrm{dz})^2 - \frac{1}{2} u \, \mathrm{dz} \, \mathrm{d}t + \frac{1}{2} \left(\frac{1}{4} u \right) (\mathrm{d}t)^2 \\
&= u \, \mathrm{dz} - \frac{1}{2} u \, \mathrm{d}t + \frac{1}{2} u \, \mathrm{d}t \\
&= u \, \mathrm{dz}.
\end{aligned}
$$

Logarithm

$$\begin{aligned}
\mathrm{d}\ln x &= \frac{\mathrm{d}\ln x}{\mathrm{d}x}\mathrm{d}x + \frac{1}{2}\frac{\mathrm{d}^2\ln x}{\mathrm{d}x^2}(\mathrm{d}x)^2 \\
&= \left(\frac{1}{x}\right)\mathrm{d}x + \frac{1}{2}\left(-\frac{1}{x^2}\right)(\mathrm{d}x)^2 \\
&= \frac{\mathrm{d}x}{x} - \frac{1}{2}\left(\frac{\mathrm{d}x}{x}\right)^2.
\end{aligned}$$

Hence the change $\mathrm{d}\ln x$ in the logarithm is *not* the growth rate $\mathrm{d}x/x$, unless the instantaneous stochastic part of $\mathrm{d}x$ is zero.

Inverse

We have

$$(1-\mathrm{d}x)^{-1} = 1 + \mathrm{d}x + (\mathrm{d}x)^2,$$

as

$$(1-\mathrm{d}x)\left[1 + \mathrm{d}x + (\mathrm{d}x)^2\right] = 1.$$

Product Rule

$$
\begin{aligned}
\mathrm{d}(xy) &= \frac{\partial (xy)}{\partial x}\mathrm{d}x + \frac{\partial (xy)}{\partial y}\mathrm{d}y \\
&\quad + \frac{1}{2}\frac{\partial^2 (xy)}{\partial x^2}(\mathrm{d}x)^2 + \frac{\partial^2 (xy)}{\partial x \partial y}\mathrm{d}x\,\mathrm{d}y + \frac{1}{2}\frac{\partial^2 (xy)}{\partial y^2}(\mathrm{d}y)^2 \\
&= y\,\mathrm{d}x + x\,\mathrm{d}y + 0\,(\mathrm{d}x)^2 + 1\,\mathrm{d}x\,\mathrm{d}y + 0\,(\mathrm{d}y)^2 \\
&= y\,\mathrm{d}x + x\,\mathrm{d}y + \mathrm{d}x\,\mathrm{d}y.
\end{aligned}
$$

Compared to non-stochastic calculus, $\mathrm{d}x\,\mathrm{d}y$ is an extra term.

Error Rule

We prove the fundamental error rule $(\mathrm{d}z)^2 = \mathrm{d}t$, by taking the limit of the discrete-time analogue. Divide the time interval from zero to t into n periods of length Δt, so $t = n\Delta t$. Holding t fixed, define

$$\int_0^t (\mathrm{d}z)^2 := \lim_{\Delta t \to 0} \sum_{i=1}^{n} \left[\Delta z_{(i-1)\Delta t}\right]^2 .$$

Defining $e_i := \Delta z_{(i-1)\Delta t} = z_{i\Delta t} - z_{(i-1)\Delta t}$, we can restate this equation as

$$\int_0^t (\mathrm{d}z)^2 := \lim_{n \to \infty} \left(e_1^2 + \cdots + e_n^2\right) .$$

We rewrite the sum of the squared errors as

$$e_1^2 + e_2^2 + \cdots + e_n^2 = t \left\{ \frac{1}{n} \left[\left(\frac{e_1^2}{\Delta t} \right) + \left(\frac{e_2^2}{\Delta t} \right) + \cdots + \left(\frac{e_n^2}{\Delta t} \right) \right] \right\}.$$

Holding $t = n\Delta t$ fixed, take the limit as $\Delta t \to 0$, $n \to \infty$.

The expression in braces is the sample mean of n independent $\chi^2(1)$ variables. By the law of large numbers, the sample mean converges to the true mean 1 as the sample size increases. Hence

$$\lim_{n\to\infty} \left(e_1^2 + e_2^2 + \cdots + e_n^2 \right) = t.$$

Therefore

$$\int_0^t (\mathrm{d}z)^2 = t,$$

regardless of t. Of course

$$\int_0^t \mathrm{d}t = t.$$

Comparing the two integrals proves

$$(\mathrm{d}z)^2 = \mathrm{d}t.$$

Time Rule

We next prove $(\mathrm{d}t)^2 = 0$. Divide the time interval from zero to t into n periods of length Δt, so $t = n\Delta t$. By definition,

$$\begin{aligned}
\int_0^t (\mathrm{d}t)^2 &:= \lim_{\Delta t \to 0} \sum^n (\Delta t)^2 \\
&= \lim_{\Delta t \to 0} \left[n (\Delta t)^2 \right] \\
&= \lim_{\Delta t \to 0} (n\Delta t) \lim_{\Delta t \to 0} \Delta t \\
&= t0 \\
&= 0,
\end{aligned}$$

and the result follows.

Cross-Product Rule

The rule $\mathrm{d}z\,\mathrm{d}t = 0$ can be shown by a similar limiting argument.