Cost/Payoff Efficiency

Definition 1 (Cost/Payoff Efficiency) *A portfolio $\overline{\mathbf{x}}$ is cost/payoff efficient if there is no portfolio $\mathbf{x}$ such that*

$$\begin{bmatrix} -\boldsymbol{v}^\top \\ A \end{bmatrix} \boldsymbol{x} \succ \begin{bmatrix} -\boldsymbol{v}^\top \\ A \end{bmatrix} \overline{\boldsymbol{x}}.$$

Cost/payoff efficiency is Pareto efficiency: no portfolio is at least as low in cost and at least as high in payoff and also better in some aspect.

No Arbitrage

Condition 2 (No Arbitrage) *There is no opportunity for profitable arbitrage if and only if the* $\mathbf{0}$ *portfolio is cost/payoff efficient.*

If there exists some portfolio $\boldsymbol{x}$ such that

$$\begin{bmatrix} -\boldsymbol{v}^\top \\ A \end{bmatrix} \boldsymbol{x} \succ \mathbf{0},$$

then the portfolio gives an arbitrage profit. The absence of arbitrage means that no such portfolio exists.

For example, there can be no portfolio with negative cost having payoff zero in every state; to buy the portfolio would furnish an arbitrage profit.

And there can be no portfolio with zero cost having positive payoff in some state and zero payoff in the other states.

Optimization

The **0** portfolio is cost/payoff efficient if and only if the optimum value is zero in the following linear program.

Problem 3 (Arbitrage)

$$\sup_{\boldsymbol{x}} \left(\begin{bmatrix} 1 \\ \mathbf{1} \end{bmatrix}^\top \begin{bmatrix} -\boldsymbol{v}^\top \\ A \end{bmatrix} \boldsymbol{x} \right),$$

subject to the constraint

$$\begin{bmatrix} -\boldsymbol{v}^\top \\ A \end{bmatrix} \boldsymbol{x} \succeq \mathbf{0}.$$

Optimum Value

As $\boldsymbol{x} = \mathbf{0}$ satisfies the constraints, the supremum is greater than or equal to zero.

If arbitrage is possible, then there exists some portfolio $\boldsymbol{x}$ satisfying the constraints that gives a positive value for the objective function. As this portfolio can be scaled up by any arbitrary positive multiple, the supremum of the arbitrage problem is ∞.

If the maximum is zero, then there is no opportunity for arbitrage.

Lagrangian

Solve the linear program (3) via the Lagrangian

$$L(\boldsymbol{x}, a, \boldsymbol{b}) = \begin{bmatrix} 1 \\ \mathbf{1} \end{bmatrix}^{\top} \begin{bmatrix} -\boldsymbol{v}^{\top} \\ A \end{bmatrix} \boldsymbol{x} + \begin{bmatrix} a \\ \boldsymbol{b} \end{bmatrix}^{\top} \begin{bmatrix} -\boldsymbol{v}^{\top} \\ A \end{bmatrix} \boldsymbol{x},$$

in which a and $\boldsymbol{b}$ are Lagrange multipliers for the inequality constraints.

Necessary and sufficient conditions for an optimum are

$$\frac{\partial L}{\partial \boldsymbol{x}} = \begin{bmatrix} -\boldsymbol{v}^\top \\ A \end{bmatrix}^\top \begin{bmatrix} 1+a \\ \mathbf{1}+\boldsymbol{b} \end{bmatrix} = \mathbf{0},$$

together with the complementary-slackness conditions

$$\begin{bmatrix} -\boldsymbol{v}^\top \\ A \end{bmatrix} \boldsymbol{x} \succeq \mathbf{0}, \begin{bmatrix} a \\ \boldsymbol{b} \end{bmatrix} \succeq \mathbf{0}, \left\langle \begin{bmatrix} a \\ \boldsymbol{b} \end{bmatrix}, \begin{bmatrix} -\boldsymbol{v}^\top \\ A \end{bmatrix} \boldsymbol{x} \right\rangle = 0.$$

Hence $\boldsymbol{x} = \mathbf{0}$ is an optimum if and only if

$$\boldsymbol{v} = A^\top (\mathbf{1} + \boldsymbol{b}) / (1 + a)$$

$$\begin{bmatrix} a \\ \boldsymbol{b} \end{bmatrix} \succeq \mathbf{0}$$

for some a and $\boldsymbol{b}$.

Restatement

An equivalent restatement of these conditions is the following. Define

$$\boldsymbol{y} := (\mathbf{1}+\boldsymbol{b})\,/\,(1+a) \gg \mathbf{0}.$$

Then the optimum is zero or ∞ depending just on whether

$$\boldsymbol{v} = A^\top \boldsymbol{y}$$

$$\boldsymbol{y} \gg \mathbf{0}$$

has a solution. If a solution exists, then the optimum value is zero, and there is no opportunity for arbitrage. If no such solution exists, then the supremum is ∞, and there is an opportunity for arbitrage.

Fundamental Theorem of Finance

The following theorem—commonly called the "fundamental theorem of finance"—summarizes the situation.

Theorem 4 (Fundamental Theorem of Finance) *There is no opportunity for arbitrage if and only if there exists a strictly positive stochastic discount factor.*

Summary

Note that the no-arbitrage theorem was not just “pulled out of a hat.” The no-arbitrage condition was restated via the arbitrage optimization problem (3). The fundamental theorem of finance then jumps out when one solves problem.

The Law of One Price

The law of one price is necessary for no arbitrage, but not sufficient. The payoff on one portfolio might dominate the payoff on another portfolio, and yet both have the same cost. There exists an opportunity for profitable arbitrage, even though a stochastic discount factor may exist.

Although the law of one price guarantees the existence of a stochastic discount factor, the stochastic discount factor need not be strictly positive.