Return

Working in a small-risk context, we derive a first-order condition for optimum portfolio choice.

Let \( da \) denote the return on the optimum portfolio—the return that maximizes expected utility. A one-dollar investment at time \( t \) is worth \( 1 + da \) dollars at time \( t + dt \).

Let \( da_i \) denote the return on asset \( i \).

Portfolio Variation

Consider an investment of the fraction \( f \) of wealth in asset \( i \), and the fraction \( 1 - f \) in the optimum portfolio. The return on this portfolio is

\[
d_{af} := f da_i + (1 - f) da.
\]

If the investment at time \( t \) is \( w_t \), then wealth at time \( t + dt \) is

\[
w_{t+dt} = w_t [1 + (f da_i + (1 - f) da)].
\]

First-Order Condition

Theorem 1 (First-Order Condition) (Arrow [1]) For asset \( i \), the first-order condition for utility-maximizing portfolio choice is

\[
0 = E_t [u'(w_{t+dt}) (da_i - da)].
\]  

The product of the marginal utility and the difference in return has expected value zero.

Proof

For asset \( i \), the first-order condition for utility maximization is

\[
0 = \frac{d}{df} (E_t [u(w_{t+dt})]),
\]

at \( f = 0 \).
State-Dependent Utility

The result is very general. In particular, it does not require that utility depend solely on end-of-period wealth; utility might be state-dependent. One might write \( u(w_{t+d} + rt + dt) \) to make this dependence explicit.

No State Dependence

If utility depends only on wealth and is not state dependent, then the expression in the first-order condition is

\[
0 = E_t \left[ (1 - \alpha da) (da_i - da) \right]
\]

\[
= \left[ E_t (da_i) - E_t (da) \right] - \alpha da (da_i - da).
\]

The sign of the expected value in (2) determines whether higher investment in asset \( i \) increases or decreases expected utility.

Mean/Variance

In the small-risk context, we know that expected utility maximization reduces to maximizing a linear function of mean and variance. Therefore let us also derive corollary 2 in this mean/variance framework.

Expected Utility

\[
E_t [u(w_{t+d})]
\]

\[
= E_t (da_f) - \frac{1}{2} \alpha \text{Var}_t (da_f)
\]

\[
= E_t [f da_i + (1 - f) da]
\]

\[
- \frac{1}{2} \alpha \text{Var}_t [f da_i + (1 - f) da]
\]

\[
= f E_t (da_i) + (1 - f) E_t (da)
\]

\[
- \frac{1}{2} \alpha [f^2 (da_i)^2 + (1 - f)^2 (da)^2 + 2f (1 - f) da_i da].
\]
First-Order Condition

The first-order condition for a maximum is
\[
0 = \frac{d}{df} (E_t [u(w_t + \Delta t)])
\]
\[
= E_t (\Delta a) - E_t (\Delta a) - \frac{1}{2} \alpha \left[ 2f (\Delta a)^2 - 2 (1 - f) (\Delta a)^2 + 2 (1 - 2f) \Delta a \Delta a \right]
\]
\[
= E_t (\Delta a) - E_t (\Delta a) - \alpha \Delta a (\Delta a - \Delta a), \text{ at } f = 0,
\]
which yields corollary 2.

Portfolio Choice

We use the first-order condition (2) to derive optimum portfolio choice. Let
\[
r \Delta t
\]
denote the return on a risk-free asset. Let
\[
\Delta x = m \Delta t + \Delta z
\]
denote a vector of excess returns on risky assets. Here z is Wiener-Brownian motion, with non-singular variance
\[
\text{Var} (\Delta z) = V \Delta t.
\]

First-Order Condition

Written as a vector, the first-order condition (2) is
\[
0 = E_t \left\{ \left( \Delta x - 1f^\top \Delta x \right) \left[ 1 - \alpha \left( r \Delta t + f^\top \Delta x \right) \right] \right\}
\]
\[
= \left( 1 - 1f^\top \right) \left[ E_t (\Delta x) - \alpha \Delta x (\Delta x^\top) f \right] \Delta t
\]
\[
= \left( 1 - 1f^\top \right) (m - \alpha Vf) \Delta t.
\]
Evidently
\[
f = \frac{1}{\alpha} V^{-1} m
\]
is a solution, in agreement with the result via the separation theorem.

References
