Return

Working in a small-risk context, we derive a first-order condition for optimum portfolio choice.

Let $\mathrm{d}a$ denote the return on the optimum portfolio—the return that maximizes expected utility. A one-dollar investment at time t is worth $1+\mathrm{d}a$ dollars at time $t+\mathrm{d}t$.

Let $\mathrm{d}a_i$ denote the return on asset i.

Portfolio Variation

Consider an investment of the fraction f of wealth in asset i, and the fraction $1-f$ in the optimum portfolio. The return on this portfolio is

$$\mathrm{d}a_f := f\,\mathrm{d}a_i + (1-f)\,\mathrm{d}a.$$

If the investment at time t is w_t, then wealth at time $t+\mathrm{d}t$ is

$$w_{t+\mathrm{d}t} = w_t\left[1 + (f\,\mathrm{d}a_i + (1-f)\,\mathrm{d}a)\right].$$

Utility

Utility at time t is $u\left(w_{t+\mathrm{d}t}\right)$.

By definition, the expected utility

$$\mathrm{E}_t\left[u\left(w_{t+\mathrm{d}t}\right)\right]$$

is maximized when $f = 0$.

First-Order Condition

Theorem 1 (First-Order Condition) *(Arrow [1]) For asset i, the first-order condition for utility-maximizing portfolio choice is*

$$0 = \mathrm{E}_t \left[u' \left(w_{t+\mathrm{d}t} \right) \left(\mathrm{d}a_i - \mathrm{d}a \right) \right]. \tag{1}$$

The product of the marginal utility and the difference in return has expected value zero.

Proof

For asset i, the first-order condition for utility maximization is

$$0 = \frac{\mathrm{d}}{\mathrm{d}f}\left(\mathrm{E}_t\left[u\left(w_{t+\mathrm{d}t}\right)\right]\right),$$

at $f = 0$.

We evaluate

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}f}\left\{\mathrm{E}_t\left[u\left(w_{t+\mathrm{d}t}\right)\right]\right\} \\
&= \mathrm{E}_t\left[u'\left(w_{t+\mathrm{d}t}\right)\frac{\mathrm{d}}{\mathrm{d}f}\left(w_{t+\mathrm{d}t}\right)\right] \\
&= \mathrm{E}_t\left[u'\left(w_{t+\mathrm{d}t}\right)\frac{\mathrm{d}}{\mathrm{d}f}\left(w_t\left\{1+\left[f\,\mathrm{d}a_i+(1-f)\,\mathrm{d}a\right]\right\}\right)\right] \\
&= \mathrm{E}_t\left[w_t u'\left(w_{t+\mathrm{d}t}\right)\left(\mathrm{d}a_i-\mathrm{d}a\right)\right],
\end{aligned}$$

and theorem 1 follows. The sign of the expected value determines whether higher investment in asset i increases or decreases expected utility.

State-Dependent Utility

The result is very general. In particular, it does not require that utility depend solely on end-of-period wealth; utility might be state-dependent. One might write $u\left(w_{t+\mathrm{d}t}, s_{t+\mathrm{d}t}\right)$ to make this dependence explicit.

No State Dependence

If utility depends only on wealth and is not state dependent, then the expression in the first-order condition is

$$u'(w_{t+\mathrm{d}t})(\mathrm{d}a_i - \mathrm{d}a)$$
$$= \left[u'(w_t) + u''(w_t)\,\mathrm{d}w_t + \frac{1}{2}u'''(w_t)(\mathrm{d}w_t)^2\right](\mathrm{d}a_i - \mathrm{d}a)$$

$$
\begin{aligned}
&= \left[u'(w_t) + u''(w_t)\,\mathrm{d}w_t + \frac{1}{2}u'''(w_t)(\mathrm{d}w_t)^2\right](\mathrm{d}a_i - \mathrm{d}a) \\
&= \left[u'(w_t) + u''(w_t)\,\mathrm{d}w_t\right](\mathrm{d}a_i - \mathrm{d}a) \\
&= \left[u'(w_t) + u''(w_t)\,w_t\,\mathrm{d}a\right](\mathrm{d}a_i - \mathrm{d}a) \\
&= u'(w_t)\left[1 + \frac{u''(w_t)\,w_t}{u'(w_t)}\mathrm{d}a\right](\mathrm{d}a_i - \mathrm{d}a) \\
&= u'(w_t)(1 - \alpha\,\mathrm{d}a)(\mathrm{d}a_i - \mathrm{d}a).
\end{aligned}
$$

Here α is the relative risk aversion.

Setting the expected value to zero yields the following corollary to theorem 1.

Corollary 2 (No State Dependence) *If utility is not state dependent, then for asset i the first-order condition for utility-maximizing portfolio choice is*

$$
\begin{aligned}
0 &= \mathrm{E}_t\left[(1-\alpha\,\mathrm{d}a)\left(\mathrm{d}a_i-\mathrm{d}a\right)\right] \\
&= \left[\mathrm{E}_t\left(\mathrm{d}a_i\right)-\mathrm{E}_t\left(\mathrm{d}a\right)\right]-\alpha\,\mathrm{d}a\left(\mathrm{d}a_i-\mathrm{d}a\right).
\end{aligned}
\tag{2}
$$

The sign of the expected value in (2) determines whether higher investment in asset i increases or decreases expected utility.

Mean/Variance

In the small-risk context, we know that expected utility maximization reduces to maximizing a linear function of mean and variance. Therefore let us also derive corollary 2 in this mean/variance framework.

Expected Utility

$$
\begin{aligned}
&\mathrm{E}_t\left[u\left(w_{t+\mathrm{d}t}\right)\right] \\
&\quad = \mathrm{E}_t\left(\mathrm{d}a_f\right) - \frac{1}{2}\alpha \mathrm{Var}_t\left(\mathrm{d}a_f\right) \\
&\quad = \mathrm{E}_t\left[f\,\mathrm{d}a_i + (1-f)\,\mathrm{d}a\right] \\
&\quad - \frac{1}{2}\alpha \mathrm{Var}_t\left[f\,\mathrm{d}a_i + (1-f)\,\mathrm{d}a\right] \\
&\quad = f\mathrm{E}_t\left(\mathrm{d}a_i\right) + (1-f)\,\mathrm{E}_t\left(\mathrm{d}a\right) \\
&\quad - \frac{1}{2}\alpha\left[f^2\left(\mathrm{d}a_i\right)^2 + (1-f)^2\left(\mathrm{d}a\right)^2 + 2f(1-f)\;\mathrm{d}a_i\,\mathrm{d}a\right].
\end{aligned}
$$

First-Order Condition

The first-order condition for a maximum is

$$
\begin{aligned}
0 &= \frac{\mathrm{d}}{\mathrm{d}f}\left(\mathrm{E}_t\left[u\left(w_{t+\mathrm{d}t}\right)\right]\right) \\
&= \mathrm{E}_t\left(\mathrm{d}a_i\right) - \mathrm{E}_t\left(\mathrm{d}a\right) \\
&- \frac{1}{2}\alpha\left[2f\left(\mathrm{d}a_i\right)^2 - 2\left(1-f\right)\left(\mathrm{d}a\right)^2 + 2\left(1-2f\right)\,\mathrm{d}a_i\,\mathrm{d}a\right] \\
&= \mathrm{E}_t\left(\mathrm{d}a_i\right) - \mathrm{E}_t\left(\mathrm{d}a\right) - \alpha\,\mathrm{d}a\left(\mathrm{d}a_i - \mathrm{d}a\right),\ \text{at } f=0,
\end{aligned}
$$

which yields corollary 2.

Portfolio Choice

We use the first-order condition (2) to derive optimum portfolio choice. Let

$$r\,\mathrm{d}t$$

denote the return on a risk-free asset. Let

$$\mathrm{d}\boldsymbol{x} = \boldsymbol{m}\,\mathrm{d}t + \mathrm{d}\boldsymbol{z}$$

denote a vector of excess returns on risky assets. Here $\boldsymbol{z}$ is Wiener-Brownian motion, with non-singular variance

$$\mathrm{Var}\,(\mathrm{d}\boldsymbol{z}) = \boldsymbol{V}\,\mathrm{d}t\,.$$

Define the vector $\boldsymbol{f}$ as the fraction of wealth invested in the risky assets, and $1-\mathbf{1}^\top \boldsymbol{f}$ is the fraction of wealth invested in the risk-free asset.

We find the first-order condition for the optimum portfolio choice $\boldsymbol{f}$.

The vector of asset returns is

$$r\mathbf{1}\,\mathrm{d}t + \mathrm{d}\boldsymbol{x}.$$

The return on the portfolio is

$$\mathrm{d}a = r\,\mathrm{d}t + \boldsymbol{f}^\top \mathrm{d}\boldsymbol{x}.$$

First-Order Condition

Written as a vector, the first-order condition (2) is

$$\begin{aligned}\mathbf{0} &= \mathrm{E}_t\left\{\left(\mathrm{d}\boldsymbol{x} - \mathbf{1}\boldsymbol{f}^\top \mathrm{d}\boldsymbol{x}\right)\left[1 - \alpha\left(r\,\mathrm{d}t + \boldsymbol{f}^\top \mathrm{d}\boldsymbol{x}\right)\right]\right\} \\ &= \left(\mathsf{I} - \mathbf{1}\boldsymbol{f}^\top\right)\left[\mathrm{E}_t\left(\mathrm{d}\boldsymbol{x}\right) - \alpha\,\mathrm{d}\boldsymbol{x}\left(\mathrm{d}\boldsymbol{x}^\top\right)\boldsymbol{f}\right]\mathrm{d}t \\ &= \left(\mathsf{I} - \mathbf{1}\boldsymbol{f}^\top\right)\left(\boldsymbol{m} - \alpha V \boldsymbol{f}\right)\mathrm{d}t.\end{aligned}$$

Evidently

$$\boldsymbol{f} = \frac{1}{\alpha} V^{-1} \boldsymbol{m}$$

is a solution, in agreement with the result via the separation theorem.

References

[1] Kenneth J. Arrow. The theory of risk aversion. In *Individual Choice under Certainty and Uncertainty, collected papers of Kenneth J. Arrow*, pages 147–171. Harvard University Press, Cambridge, MA, 1984. HD30.23A74 1984. 1