Return

Working in a small-risk context, we derive a first-order condition for optimum portfolio choice.

Let $da$ denote the return on the optimum portfolio—the return that maximizes expected utility. A one-dollar investment at time $t$ is worth $1 + da$ dollars at time $t + dt$.

Let $da_i$ denote the return on asset $i$. 
Portfolio Variation

Consider an investment of the fraction $f$ of wealth in asset $i$, and the fraction $1 - f$ in the optimum portfolio. The return on this portfolio is

$$d a_f := f d a_i + (1 - f) d a.$$ 

If the investment at time $t$ is $w_t$, then wealth at time $t + dt$ is

$$w_{t+dt} = w_t [1 + (f d a_i + (1 - f) d a)].$$
Utility

Utility at time $t$ is $u(w_{t+dt})$.

By definition, the expected utility

$$E_t [u(w_{t+dt})]$$

is maximized when $f = 0$. 
First-Order Condition

Theorem 1 (First-Order Condition) (Arrow [1]) For asset \( i \), the first-order condition for utility-maximizing portfolio choice is

\[
0 = \mathbb{E}_t \left[ u'(w_{t+dt}) (da_i - da) \right].
\]  

(1)

The product of the marginal utility and the difference in return has expected value zero.
Proof

For asset $i$, the first-order condition for utility maximization is

$$0 = \frac{d}{df} (E_t [u(w_{t+dt})]),$$

at $f = 0$. 
We evaluate

\[
\frac{d}{df} \left\{ E_t \left[ u (w_{t+dt}) \right] \right\}
\]

\[
= E_t \left[ u' (w_{t+dt}) \frac{d}{df} (w_{t+dt}) \right]
\]

\[
= E_t \left[ u' (w_{t+dt}) \frac{d}{df} (w_t \{1 + [f da_i + (1 - f) da]\}) \right]
\]

\[
= E_t \left[ w_t u' (w_{t+dt}) (da_i - da) \right],
\]

and theorem 1 follows. The sign of the expected value determines whether higher investment in asset \( i \) increases or decreases expected utility.
State-Dependent Utility

The result is very general. In particular, it does not require that utility depend solely on end-of-period wealth; utility might be state-dependent. One might write \( u(w_{t+dt}, s_{t+dt}) \) to make this dependence explicit.
No State Dependence

If utility depends only on wealth and is not state dependent, then the expression in the first-order condition is

\[ u'(w_{t+dt})(da_i - da) = \left[ u'(w_t) + u''(w_t) dw_t + \frac{1}{2} u'''(w_t)(dw_t)^2 \right] (da_i - da) \]
\[
\begin{align*}
\text{Financial Economics} & \quad \text{First-Order Condition} \\
= & \left[ u'(w_t) + u''(w_t) \, dw_t + \frac{1}{2} u'''(w_t) \, (dw_t)^2 \right] (da_i - da) \\
= & \left[ u'(w_t) + u''(w_t) \, dw_t \right] (da_i - da) \\
= & \left[ u'(w_t) + u''(w_t) \, w_t \, da \right] (da_i - da) \\
= & u'(w_t) \left[ 1 + \frac{u''(w_t) \, w_t}{u'(w_t)} \, da \right] (da_i - da) \\
= & u'(w_t) \left( 1 - \alpha \, da \right) (da_i - da).
\end{align*}
\]

Here \( \alpha \) is the relative risk aversion.
Setting the expected value to zero yields the following corollary to theorem 1.

**Corollary 2 (No State Dependence)** \( If \) utility is not state dependent, \( then \) for asset \( i \) the first-order condition for utility-maximizing portfolio choice is

\[
0 = E_t [(1 - \alpha da) (da_i - da)] \\
= [E_t (da_i) - E_t (da)] - \alpha da (da_i - da).
\]

The sign of the expected value in (2) determines whether higher investment in asset \( i \) increases or decreases expected utility.
Mean/Variance

In the small-risk context, we know that expected utility maximization reduces to maximizing a linear function of mean and variance. Therefore let us also derive corollary 2 in this mean/variance framework.
Financial Economics

Expected Utility

\[ E_t \left[ u (w_{t+dt}) \right] \]

\[ = E_t (da_f) - \frac{1}{2} \alpha \text{Var}_t (da_f) \]

\[ = E_t [f \, da_i + (1 - f) \, da] \]

\[ - \frac{1}{2} \alpha \text{Var}_t [f \, da_i + (1 - f) \, da] \]

\[ = f E_t (da_i) + (1 - f) E_t (da) \]

\[ - \frac{1}{2} \alpha \left[ f^2 (da_i)^2 + (1 - f)^2 (da)^2 + 2f (1 - f) \, da_i \, da \right]. \]
First-Order Condition

The first-order condition for a maximum is

\[ 0 = \frac{d}{df} (E_t [u(w_{t+dt})]) = E_t (da_i) - E_t (da) - \frac{1}{2} \alpha \left[ 2f (da_i)^2 - 2(1 - f)(da)^2 + 2(1 - 2f) da_i da \right] = E_t (da_i) - E_t (da) - \alpha da (da_i - da), \text{ at } f = 0, \]

which yields corollary 2.
Portfolio Choice

We use the first-order condition (2) to derive optimum portfolio choice. Let

\[ r\, dt \]

denote the return on a risk-free asset. Let

\[ dx = m\, dt + dz \]

denote a vector of excess returns on risky assets. Here \( z \) is Wiener-Brownian motion, with non-singular variance

\[ \text{Var}(dz) = V\, dt. \]
Define the vector $f$ as the fraction of wealth invested in the risky assets, and $1 - 1^T f$ is the fraction of wealth invested in the risk-free asset.

We find the first-order condition for the optimum portfolio choice $f$.

The vector of asset returns is

$$r \mathbf{1} \, dt + d\mathbf{x}.$$ 

The return on the portfolio is

$$d\mathbf{a} = r \, dt + f^T \, d\mathbf{x}.$$
First-Order Condition

Written as a vector, the first-order condition (2) is

\[
0 = E_t \left\{ \left( dx - 1f^\top dx \right) \left[ 1 - \alpha \left( r dt + f^\top dx \right) \right] \right\} \\
= \left( I - 1f^\top \right) \left[ E_t (dx) - \alpha dx \left( dx^\top \right) f \right] dt \\
= \left( I - 1f^\top \right) (m - \alpha Vf) dt.
\]

Evidently

\[
f = \frac{1}{\alpha} V^{-1} m
\]

is a solution, in agreement with the result via the separation theorem.
References
