

Von Neumann and Morgenstern Expected Utility Maximization

Define a utility function so choice under uncertainty maximizes the expected utility of wealth,

$$E[u(w)].$$

We assume positive marginal utility.

1

Utility Unique Only up to Positive Linear Transformation

For

$$v(w) = a + bu(w), b > 0,$$

then

$$E[v(w)] = a + bE[u(w)],$$

so the two utility functions are equivalent.

2

Risk Indifference

Risk indifference means that the individual chooses the gamble to maximize expected wealth

$$E(w).$$

The individual is risk indifferent if and only if the utility function is linear,

$$u(w) = a + bw, b > 0,$$

so

$$E[u(w)] = a + bE(w).$$

3

Risk Aversion

The individual is risk averse if he will trade off less risk for a reduced expected value. The individual is risk averse if and only if the utility function is concave.

4

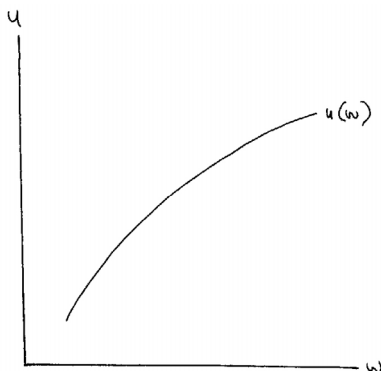


Figure 1: Risk Aversion

5

Axiomatic Basis

If the choice under uncertainty satisfies certain reasonable axioms, then one can construct a utility function that explains the choice (von Neumann and Morgenstern [1]).

Axioms:

- Completeness;
- Transitivity;
- Continuity;
- Substitution.

6

Completeness

For any two gambles A and B, the individual either prefers one to the other or is indifferent between them: either $A \succ B$, $B \succ A$, or $A \sim B$.

7

Transitivity

$A \succ B$ and $B \succ C$ implies $A \succ C$.

8

Continuity

Notation: $p \circ A + (1 - p) \circ B$ means to receive A with probability p and B with probability $1 - p$.

Continuity: if $A \succ B \succ C$, then there exists some probability p such that

$$B \sim p \circ A + (1 - p) \circ C.$$

9

Substitution

The substitution of indifferent gambles has no effect on the preference ordering:

$$p \circ A + (1 - p) \circ C \sim p \circ B + (1 - p) \circ C$$

if $A \sim B$.

10

Example of Substitution

A: \$5 with probability 1.

B: \$10 with probability 1.

C: \$4 with probability $\frac{1}{2}$, \$7 with probability $\frac{1}{2}$.

If $A \sim C$, then the individual is indifferent between:

\$5 with probability p , \$10 with probability $1 - p$.

\$4 with probability $\frac{1}{2}p$, \$7 with probability $\frac{1}{2}p$, \$10 with probability $1 - p$.

11

Definition of Utility

Let A denote the best outcome, and let Z denote the worst outcome. For a gamble B, define its utility $u(B)$ as the probability such that

$$B \sim p \circ A + (1 - p) \circ Z.$$

12

Proof of Expected Utility Property

To show:

$$u[p \circ B + (1 - p) \circ C] = pu(B) + (1 - p)u(C).$$

Proof:

$$\begin{aligned} p \circ B + (1 - p) \circ C &\sim p \circ [u(B) \circ A + (1 - u(B)) \circ Z] \\ &\quad + (1 - p) \circ [u(C) \circ A + (1 - u(C)) \circ Z] \\ &\sim [pu(B) + (1 - p)u(C)] \circ A \\ &\quad + \{p[1 - u(B)] + (1 - p)[1 - u(C)]\} \circ Z. \end{aligned}$$

The latter has utility $pu(B) + (1 - p)u(C)$.

References

- [1] J. von Neumann and O. Morgenstern. *Theory of Games and Economic Behavior*. Princeton University Press, Princeton, NJ, third edition, 1953. QA269V65 1953 (originally 1944).