Von Neumann and Morgenstern Expected Utility Maximization

Define a utility function so choice under uncertainty maximizes the expected utility of wealth,

$$\mathrm{E}\left[u\left(w\right)\right].$$

We assume positive marginal utility.

Utility Unique Only up to Positive Linear Transformation

For

$$v(w) = a + bu(w), b > 0,$$

then

$$\mathrm{E}[v(w)] = a + b\mathrm{E}[u(w)],$$

so the two utility functions are equivalent.

Risk Indifference

Risk indifference means that the individual chooses the gamble to maximize expected wealth

$$\mathrm{E}\left(w\right).$$

The individual is risk indifferent if and only if the utility function is linear,

$$u\left(w\right)=a+bw, b>0,$$

so

$$\mathrm{E}\left[u\left(w\right)\right]=a+b\mathrm{E}\left(w\right).$$

Risk Aversion

The individual is risk averse if he will trade off less risk for a reduced expected value. The individual is risk averse if and only if the utility function is concave.

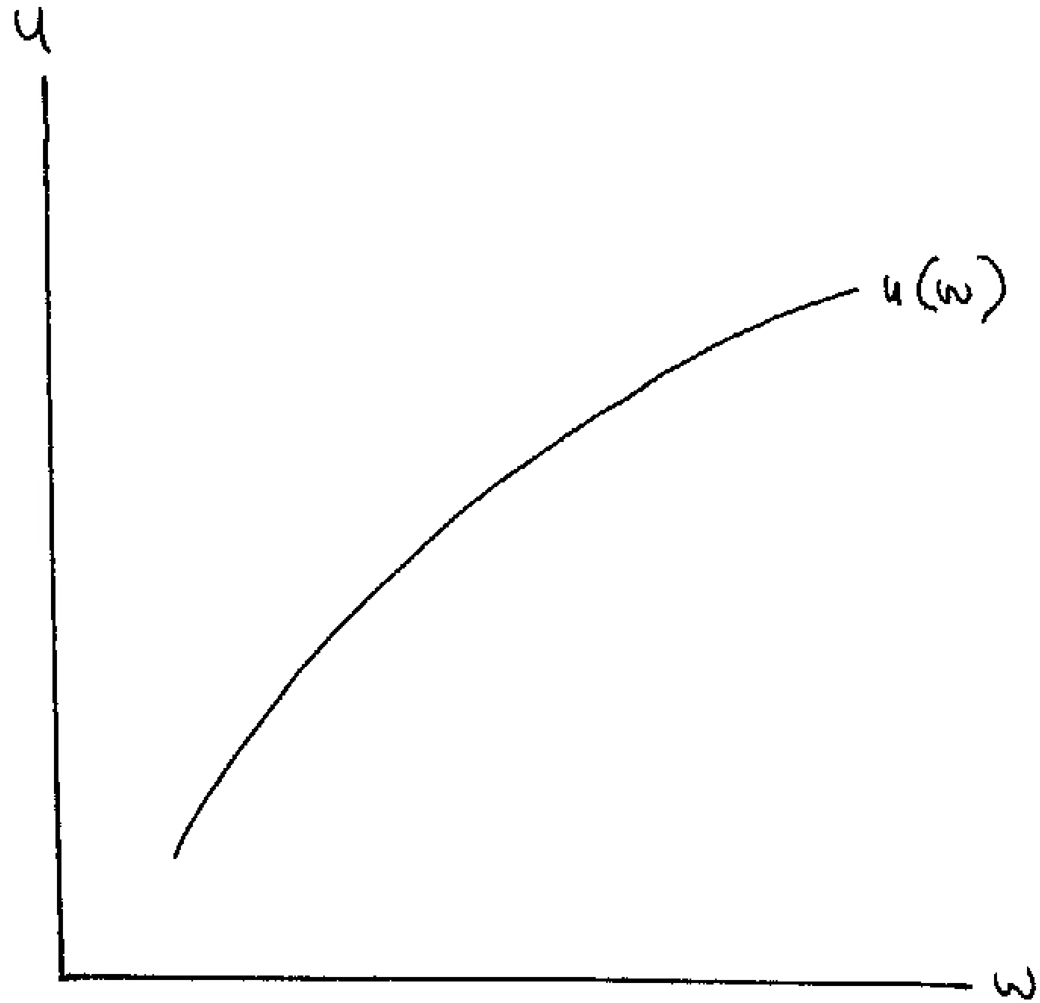

Figure 1: Risk Aversion

Axiomatic Basis

If the choice under uncertainty satisfies certain reasonable axioms, then one can construct a utility function that explains the choice (von Neumann and Morgenstern [1]).

Axioms:

- Completeness;
- Transitivity;
- Continuity;
- Substitution.

Completeness

For any two gambles A and B, the individual either prefers one to the other or is indifferent between them: either A $\succ$ B, B $\succ$ A, or A $\sim$ B.

Transitivity

A $\succ$ B and B $\succ$ C implies A $\succ$ C.

Continuity

Notation: $p \circ \mathrm{A} + (1-p) \circ \mathrm{B}$ means to receive A with probability p and B with probability $1-p$.

Continuity: if $\mathrm{A} \succ \mathrm{B} \succ \mathrm{C}$, then there exists some probability p such that

$$\mathrm{B} \sim p \circ \mathrm{A} + (1-p) \circ \mathrm{C}.$$

Substitution

The substitution of indifferent gambles has no effect on the preference ordering:

$$p \circ \mathrm{A} + (1-p) \circ \mathrm{C} \sim p \circ \mathrm{B} + (1-p) \circ \mathrm{C}$$

if A $\sim$ B.

Example of Substitution

A: \$5 with probability 1.

B: \$10 with probability 1.

C: \$4 with probability $\frac{1}{2}$, \$7 with probability $\frac{1}{2}$.

If A $\sim$ C, then the individual is indifferent between:

\$5 with probability p, \$10 with probability $1-p$.

\$4 with probability $\frac{1}{2}p$, \$7 with probability $\frac{1}{2}p$, \$10 with probability $1-p$.

Definition of Utility

Let A denote the best outcome, and let Z denote the worst outcome. For a gamble B, define its utility $u(\mathrm{B})$ as the probability such that

$$\mathrm{B} \sim p \circ \mathrm{A} + (1-p) \circ \mathrm{Z}.$$

Proof of Expected Utility Property

To show:

$$u\left[p \circ \mathrm{B} + (1-p) \circ \mathrm{C}\right] = pu(\mathrm{B}) + (1-p)\,u(\mathrm{C})\,.$$

Proof:

$$\begin{aligned} p \circ \mathrm{B} + (1-p) \circ \mathrm{C} &\sim p \circ \left[u(\mathrm{B}) \circ \mathrm{A} + (1-u(\mathrm{B})) \circ \mathrm{Z}\right] \\ &\quad + (1-p) \circ \left[u(\mathrm{C}) \circ \mathrm{A} + (1-u(\mathrm{C})) \circ \mathrm{Z}\right] \\ &\sim \left[pu(\mathrm{B}) + (1-p)\,u(\mathrm{C})\right] \circ \mathrm{A} \\ &\quad + \{p\left[1-u(\mathrm{B})\right] + (1-p)\left[1-u(\mathrm{C})\right]\} \circ \mathrm{Z}. \end{aligned}$$

The latter has utility $pu(\mathrm{B}) + (1-p)\,u(\mathrm{C})$.

References

[1] J. von Neumann and O. Morgenstern. *Theory of Games and Economic Behavior*. Princeton University Press, Princeton, NJ, third edition, 1953. QA269V65 1953 (originally 1944).