Euclidean Space

Definition 1 (Euclidean Space) *A Euclidean space is a finite-dimensional vector space over the reals* $\mathbf{R}$*, with an inner product* $\langle \cdot, \cdot \rangle$.

Inner Product

Definition 2 (Inner Product) *An inner product* $\langle \cdot, \cdot \rangle$ *on a real vector space* $\mathbf{X}$ *is a symmetric, bilinear, positive-definite function*

$$\langle \cdot, \cdot \rangle : \mathbf{X} \times \mathbf{X} \to \mathbf{R}$$
$$(\boldsymbol{x}^*, \boldsymbol{x}) \mapsto \langle \boldsymbol{x}^*, \boldsymbol{x} \rangle .$$

(Positive-definite means $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0$ *unless* $\boldsymbol{x} = \mathbf{0}$*.)*

Orthogonal

Definition 3 (Orthogonal) *Two vectors* $\boldsymbol{x}^*$ *and* $\boldsymbol{x}$ *are orthogonal if their inner product is zero,*

$$\langle \boldsymbol{x}^*, \boldsymbol{x} \rangle = 0.$$

Geometrically, orthogonal means perpendicular.

Orthonormal Basis

Definition 4 (Orthonormal Basis) *In a Euclidean space, an orthonormal basis is a basis* $\boldsymbol{x}_i$ *such that*

$$\langle \boldsymbol{x}_i, \boldsymbol{x}_j \rangle = \begin{cases} 1 \text{ } \mathit{if} \text{ } i = j \\ 0 \text{ } \mathit{if} \text{ } i \neq j. \end{cases}$$

Any two basis vectors are orthogonal.

A Euclidean space has more than one orthonormal basis.

If

$$\boldsymbol{x} = \sum_i x_i \boldsymbol{x}_i$$

$$\boldsymbol{x}^* = \sum_i x_i^* \boldsymbol{x}_i,$$

then

$$\langle \boldsymbol{x}^*, \boldsymbol{x} \rangle = \sum_i x_i^* x_i.$$

R

For the real numbers **R**, the inner product is just ordinary multiplication.

Definition 5 *The Euclidean space* **R** *of real numbers is defined by the inner product*

$$\langle x^*, x \rangle := x^* x.$$

$\mathbf{R}^n$

The Euclidean space $\mathbf{R}^n := \mathbf{R} \times \cdots \times \mathbf{R}$ (n times), in which the elements are vectors with n real components. By assumption, the n vectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix}, \cdots.$$

form an orthonormal basis. The inner product of two vectors is then the sum of the component by component products.

Isomorphic

In abstract algebra, “isomorphic” means “the same.” If two objects of a given type (group, ring, vector space, Euclidean space, algebra, etc.) are isomorphic, then they are “the same,” when considered as objects of that type. An “isomorphism” is a one-to-one and onto mapping from one space to the other that “preserves” all properties defining the space.

Any n-dimensional Euclidean space is isomorphic to $\mathbf{R}^n$.

Although two spaces may be isomorphic as Euclidean spaces, perhaps the “same” two spaces are not isomorphic when viewed as another space.

Coordinate-Free Versus Basis

It is useful to think of a vector in a Euclidean space as coordinate-free.

Given a basis, any vector can be expressed uniquely as a linear combination of the basis elements. For example, if $\boldsymbol{x} = \sum_i x_i \boldsymbol{x}_i$ for some basis $\boldsymbol{x}_i$, one can refer to the x_i as the coordinates of $\boldsymbol{x}$ in terms of this basis. Many linear algebra textbooks develop all the results in terms of a basis.

In economic theory and econometrics, typically vectors are not seen as coordinate-free. A particular basis is singled out, and one works with coordinates. Commonly there is a natural basis, but unfortunately the natural basis is perhaps not orthonormal.

Despite this tradition, the coordinate-free point-of-view is superior. Not using coordinates reduces the use of subscripts and makes expressions simpler, and theorems are easier to state and to prove.

Linear Transformation

Definition 6 (Linear Transformation) *A linear transformation from a Euclidean space* $\mathbf{X}$ *to a Euclidean space* $\mathbf{Y}$ *is a function*

$$A : \mathbf{X} \to \mathbf{Y}$$

$$\boldsymbol{x} \mapsto \boldsymbol{y} = A\boldsymbol{x}$$

such that

$$A(\boldsymbol{x}_1 + \boldsymbol{x}_2) = A\boldsymbol{x}_1 + A\boldsymbol{x}_2.$$

Adjoint

The following proposition is a standard theorem of linear algebra.

Proposition 7 (Adjoint) *Given a linear transformation* $A : \mathbf{X} \to \mathbf{Y}$*, then there exists a unique linear transformation (the adjoint)*

$$A^\top : \mathbf{Y} \to \mathbf{X}$$

that preserves the inner product:

$$\langle \boldsymbol{y}, A\boldsymbol{x} \rangle = \left\langle A^\top \boldsymbol{y}, \boldsymbol{x} \right\rangle \tag{1}$$

for all $\boldsymbol{x}$ *and* $\boldsymbol{y}$.

The adjoint is very important in applications and has not been appreciated by economists. The adjoint is independent of any choice of bases, and in many applications one can determine it directly, expressed in a coordinate-free way. The adjoint then becomes a powerful tool, and one can easily obtain valuable results via the adjoint, almost as if by magic.

Typically one does not calculate the adjoint directly. Instead one conjectures an expression for the adjoint, and then verifies that the adjoint condition (1) holds.

Matrix Representation

A *matrix representation* for a linear transformation $A : \mathbf{X} \to \mathbf{Y}$ is a matrix A_{ij} that shows how basis elements $\boldsymbol{x}_j \in \mathbf{X}$ map to a linear combination of basis elements $\boldsymbol{y}_i \in \mathbf{Y}$:

$$\boldsymbol{x}_j \mapsto A\boldsymbol{x}_j = \sum_i A_{ij}\boldsymbol{y}_i.$$

Adjoint as Transpose

If the bases for **X** and **Y** are each orthonormal, then the matrix representation of the adjoint is the transpose of the matrix representation:

$$A^\top \boldsymbol{y}_i = \sum_j A_{ij} \boldsymbol{x}_j.$$

To prove this relationship, verify the adjoint condition (1), for arbitrary basis elements:

$$\begin{aligned}\left\langle \mathsf{A}^\top \boldsymbol{y}_i, \boldsymbol{x}_j \right\rangle &= \left\langle \sum_k A_{ik} \boldsymbol{x}_k, \boldsymbol{x}_j \right\rangle \\ &= A_{ij} \text{ (since the basis } \boldsymbol{x}_j \text{ is orthonormal)} \\ &= \left\langle \boldsymbol{y}_i, \sum_k A_{kj} \boldsymbol{y}_k \right\rangle \text{ (since the basis } \boldsymbol{y}_i \text{ is orthonormal)} \\ &= \langle \boldsymbol{y}_i, \mathsf{A}\boldsymbol{x}_j \rangle,\end{aligned}$$

as desired.

On the other hand, if the bases are not orthonormal, then the transpose of the matrix representation is *not* the matrix representation of the adjoint.

Since we want to see vectors as coordinate-free, however, the matrix representation is of secondary importance. Apart from simple cases, it may be difficult to write down the matrix representation explicitly. At the same time, one can describe the adjoint easily, without reference to any basis.

Riesz Representation

A fundamental theorem states that any linear function $\mathbf{X} \to \mathbf{R}$ can be expressed as $\boldsymbol{x} \mapsto \langle \boldsymbol{y}, \boldsymbol{x} \rangle$ for a unique $\boldsymbol{y}$.

For some $\boldsymbol{y} \in \mathbf{X}$, the adjoint of the linear function

$$\boldsymbol{y} : \mathbf{R} \mapsto \mathbf{X}$$

$$z \mapsto \boldsymbol{x} = z\boldsymbol{y}$$

is

$$\boldsymbol{y}^{\top} : \mathbf{X} \to \mathbf{R}$$

$$\boldsymbol{x} \mapsto z = \langle \boldsymbol{y}, \boldsymbol{x} \rangle .$$

Verify that the adjoint condition (1) holds:

$$\langle \boldsymbol{x}, \boldsymbol{y}z \rangle = \langle \boldsymbol{x}, \boldsymbol{y} \rangle z = \langle \boldsymbol{y}, \boldsymbol{x} \rangle z = \langle \langle \boldsymbol{y}, \boldsymbol{x} \rangle , z \rangle = \left\langle \boldsymbol{y}^\top \boldsymbol{x}, z \right\rangle .$$

Thus

$$\boldsymbol{y}^\top \boldsymbol{x} = \langle \boldsymbol{y}, \boldsymbol{x} \rangle .$$

Either notation is equivalent, but normally we employ the inner product notation on the right-hand side.

Matrix Representation

Suppose

$$\boldsymbol{y} = \sum_i y_i \boldsymbol{x}_i,$$

for a basis $\boldsymbol{x}_i$. Let us use the natural orthonormal basis 1 for $\mathbf{R}$.

The matrix representation of the linear transformation $\boldsymbol{y}$ is

$$1 \to \sum_i y_i \boldsymbol{x}_i,$$

so the vector with components y_i defines the matrix representation. For the adjoint $\boldsymbol{y}^\top$, however, the matrix representation is *not* the transpose of this vector, unless the basis $\boldsymbol{x}_i$ is orthonormal.

The matrix representation of the adjoint is

$$\boldsymbol{x}_j \mapsto \langle \boldsymbol{y}, \boldsymbol{x}_j \rangle 1 = \left\langle \sum_i y_i \boldsymbol{x}_i, \boldsymbol{x}_j \right\rangle 1 = \sum_i \langle \boldsymbol{x}_i, \boldsymbol{x}_j \rangle y_i 1.$$

For a nonorthonormal basis, the matrix representation of the adjoint is *not* $\boldsymbol{x}_j \mapsto y_j 1$.

Fundamental Theorem of Linear Algebra

The fundamental theorem of linear algebra states that the null space $\mathrm{N}(A)$ and the range $\mathrm{R}\left(A^\top\right)$ are orthogonal, and any $\boldsymbol{x} \in \mathbf{X}$ can be written uniquely as an element of $\mathrm{N}(A)$ plus an element of $\mathrm{R}\left(A^\top\right)$.

The same relationship holds for the range $\mathrm{R}(A)$ and the null space $\mathrm{N}\left(A^\top\right)$.

Moore-Penrose Generalized Inverse

Using the fundamental theorem of linear algebra, we define the Moore-Penrose generalized inverse.

Consider a linear transformation

$$A : \mathbf{X} \to \mathbf{Y}$$
$$\boldsymbol{x} \mapsto \boldsymbol{y} = A\boldsymbol{x}.$$

The generalized inverse A^+ is a linear transformation mapping $\mathbf{Y} \to \mathbf{X}$.

Using the fundamental theorem of linear algebra, one can prove the following.

Proposition 8 (Inverse) *The restriction of* A *to* $\mathrm{R}\left(A^\top\right)$

$$A : \mathrm{R}\left(A^\top\right) \to \mathrm{R}(A)$$
$$\boldsymbol{x} \mapsto \boldsymbol{y} = A\boldsymbol{x}.$$

is one-to-one and onto, so it has an inverse.

Define the generalized inverse

$$A^{+} : \mathbf{Y} \rightarrow \mathbf{X}$$

via this inverse mapping. For $\boldsymbol{y} \in \mathrm{R}(A)$, define $A^{+}\boldsymbol{y}$ as the inverse of A. For $\boldsymbol{y} \in \mathrm{N}\left(A^{\top}\right)$, define $A^{+}\boldsymbol{y} = \mathbf{0}$. This definition rests on the coordinate-free approach to linear algebra.

The relationship between A and A^{+} is symmetric. A linear transformation is the generalized inverse of its generalized inverse: $(A^{+})^{+} = A$. And $AA^{+}A = A$.

The singular-value decomposition obtains further results. If A is onto, then $AA^{\top}$ is invertible, and $A^{+} = A^{\top}\left(AA^{\top}\right)^{-1}$.

Linear Equation

For the linear equation

$$A\boldsymbol{x} = \boldsymbol{y},$$

there is a solution $\boldsymbol{x}$ if and only if $\boldsymbol{y} \in \mathrm{R}(A)$.

If there is a solution, then the unique solution in $\mathrm{R}\left(A^\top\right)$ is $A^+\boldsymbol{y}$. This vector plus any element of $\mathrm{N}(A)$ is also a solution. Hence the complete solution set is

$$\left\{A^+\boldsymbol{y}\right\} + \mathrm{N}(A).$$

Order

The concept of a Euclidean space does not involve any concept of order, of one vector being greater than another. However commonly one defines the additional structure of a partial ordering via the representation of a vector in a natural basis.

Definition 9 (Partial Ordering) *Given a basis representation*

$$\boldsymbol{x} = \sum_j x_j \boldsymbol{x}_j,$$

then

$$\boldsymbol{x} \succeq \mathbf{0} \textit{ if every } x_j \geq 0$$

$$\boldsymbol{x} \succ \mathbf{0} \textit{ if every } x_j \geq 0 \textit{ and some } x_i > 0$$

$$\boldsymbol{x} \gg \mathbf{0} \textit{ if every } x_j > 0.$$

Definition of the Inner Product

To embed model structure into the inner product simplifies the analysis.

In a Euclidean space of random variables, one might define the inner product of two random variables as the covariance. Orthogonality then means no correlation.

A different definition of the inner product derives from a partial ordering: one defines a “trace” inner product consistent with the ordering.