Efficiency

A payoff is cost/mean/variance efficient if no other payoff has the same or lower cost, the same or higher mean, and the same or lower variance, and better in at least one respect.

Payoff

Consider a portfolio with payoff $\boldsymbol{y}$ in the payoff space.

Assume that the law of one price holds, so the pricing kernel $\boldsymbol{p}$ is uniquely defined.

The cost to buy the portfolio is $c = \langle \boldsymbol{p}, \boldsymbol{y} \rangle$.

The mean value of the payoff is $m = \langle \boldsymbol{e}, \boldsymbol{y} \rangle$.

The variance of the payoff is the expected squared payoff less the square of the expected payoff,

$$\langle \boldsymbol{y}, \boldsymbol{y} \rangle - \langle \boldsymbol{e}, \boldsymbol{y} \rangle^2 = \left\langle \boldsymbol{y}, \left(\mathsf{I} - \boldsymbol{e}\boldsymbol{e}^\top \right) \boldsymbol{y} \right\rangle .$$

Linear Combination

Any payoff can be expressed uniquely in the form

$$\boldsymbol{y} = g\boldsymbol{p} + h\boldsymbol{e} + \boldsymbol{u},$$

in which $\boldsymbol{u}$ is orthogonal to both $\boldsymbol{p}$ and $\boldsymbol{e}$. (For a given $\boldsymbol{y}$, then g and h are the coefficients in the least-squares linear regression of $\boldsymbol{y}$ on $\boldsymbol{p}$ and $\boldsymbol{e}$.)

Cost

The portfolio cost is

$$c = \langle \boldsymbol{p}, g\boldsymbol{p} + h\boldsymbol{e} + \boldsymbol{u} \rangle = \langle \boldsymbol{p}, g\boldsymbol{p} + h\boldsymbol{e} \rangle \, ,$$

so the choice of $\boldsymbol{u}$ has no effect on the cost.

Mean

The mean payoff is

$$m = \langle \boldsymbol{e}, g\boldsymbol{p} + h\boldsymbol{e} + \boldsymbol{u} \rangle = \langle \boldsymbol{e}, g\boldsymbol{p} + h\boldsymbol{e} \rangle \,,$$

so the choice of $\boldsymbol{u}$ has no effect on the mean.

Variance

The payoff variance is

$$\begin{aligned} v &= \left\langle \boldsymbol{y}, \left(\mathsf{I} - \boldsymbol{e}\boldsymbol{e}^\top\right) \boldsymbol{y} \right\rangle \\ &= \left\langle g\boldsymbol{p} + h\boldsymbol{e} + \boldsymbol{u}, \left(\mathsf{I} - \boldsymbol{e}\boldsymbol{e}^\top\right) (g\boldsymbol{p} + h\boldsymbol{e} + \boldsymbol{u}) \right\rangle \\ &= \left\langle g\boldsymbol{p} + h\boldsymbol{e}, \left(\mathsf{I} - \boldsymbol{e}\boldsymbol{e}^\top\right) (g\boldsymbol{p} + h\boldsymbol{e}) \right\rangle + \langle \boldsymbol{u}, \boldsymbol{u} \rangle , \end{aligned}$$

so the choice $\boldsymbol{u} = \mathbf{0}$ minimizes the variance.

Separation Theorem

Theorem 1 (Separation) *The payoff on any cost/mean/variance efficient portfolio is a linear combination of the pricing kernel and the expectations kernel.*

Risk-Free Asset

For simplicity, let us assume that there exists a risk-free asset.

Condition 2 (Risk-Free Asset)

$$\boldsymbol{e} = \mathbf{1}.$$

Let us also assume that at least one asset has a non-zero risk premium.

Condition 3 (Risk Premium) *The pricing kernel* $\mathbf{p}$ *is not proportional to the expectations kernel* $\mathbf{1}$.

If it were proportional, then the price of every portfolio would be proportional to its expected payoff.

Condition 4 (Positive Cost for Risk-Free Asset)

$$\langle \boldsymbol{p}, \mathbf{1} \rangle > 0.$$

Of course $\langle \boldsymbol{p}, \mathbf{1} \rangle$ is the cost of a risk-free asset yielding one in each state. If this cost were negative or zero, that it has a positive return would imply an arbitrage opportunity.

Our assumptions have not ruled out every possibility for arbitrage. Cost/mean/variance efficiency does not rule out the possibility of arbitrage.

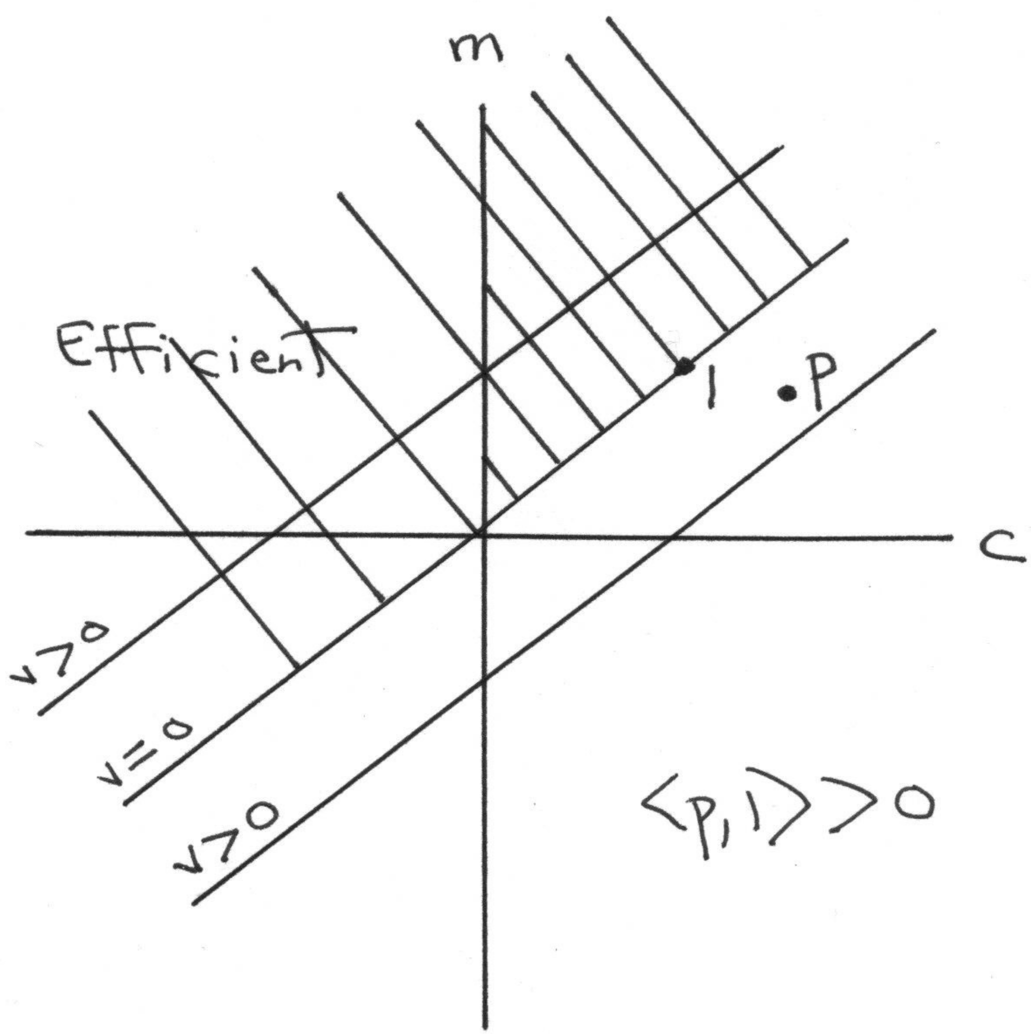
Cost/Mean Space

Since any efficient portfolio is some linear combination $g\boldsymbol{p}+h\mathbf{1}$ of $\boldsymbol{p}$ and $\mathbf{1}$, let us analyze portfolio efficiency in (c,m)-space (figure 1). For any values c and m, a unique linear combination of $\boldsymbol{p}$ and $\mathbf{1}$ yields these values.

The pricing kernel has cost $\langle\boldsymbol{p},\boldsymbol{p}\rangle>0$ and mean $\langle\boldsymbol{p},\mathbf{1}\rangle>0$. The point $\boldsymbol{p}$ in the upper right quadrant shows this cost and mean.

The expectations kernel has cost $\langle\boldsymbol{p},\mathbf{1}\rangle>0$ and mean $\langle\mathbf{1},\mathbf{1}\rangle=1$. The point $\mathbf{1}$ in the upper right quadrant shows this cost and mean.

Figure 1: Efficient Portfolios



Slope

That the variance of $\boldsymbol{p}$ is necessarily positive means

$$\langle \boldsymbol{p}, \boldsymbol{p} \rangle > \langle \boldsymbol{p}, \mathbf{1} \rangle^2 .$$

Rearranging gives

$$\frac{\langle \mathbf{1}, \mathbf{1} \rangle}{\langle \boldsymbol{p}, \mathbf{1} \rangle} > \frac{\langle \boldsymbol{p}, \mathbf{1} \rangle}{\langle \boldsymbol{p}, \boldsymbol{p} \rangle}.$$

The fraction on the left is the ratio of mean to cost for the expectations kernel. The fraction on the right is the ratio for the pricing kernel. Thus the slope from the origin to **1** is steeper than the slope to $\boldsymbol{p}$.

Variance

For payoff $g\boldsymbol{p}+h\mathbf{1}$, the variance is

$$v=\left\langle g\boldsymbol{p}+h\mathbf{1},\left(\mathsf{I}-\mathbf{1}\mathbf{1}^{\top}\right)(g\boldsymbol{p}+h\mathbf{1})\right\rangle=g^{2}\langle\boldsymbol{p},\boldsymbol{p}\rangle.$$

The variance is zero on the line from the origin passing through $\mathbf{1}$; along this line, the payoff is proportional to $\mathbf{1}$ ($g=0$). Along any parallel line, the variance is constant (g fixed).

Efficiency

The cost/mean/variance efficient points are those in the half-space on and above the line from the origin passing through **1** (always $g \leq 0$).

Note that some efficient points have negative mean or negative cost. The investor can choose a negative cost if he will accept a negative mean, or a positive mean with a higher variance.

The points relevant for market equilibrium are those with positive cost, the points in the more shaded cone in the upper right quadrant ($h > 0$). Moving upward vertically gains a higher mean, but offset by higher variance.

Portfolio Choice

Consider an individual who maximizes utility, a function of cost, mean, and variance,

$$u(c,m,v), u_c < 0, u_m > 0, u_v < 0.$$

When risks are small, the tradeoff of mean and variance is half the absolute risk aversion, so let us use the notation

$$a := -2\frac{u_v}{u_m}.$$

Pricing Kernel and Expectations Kernel

Expressed in terms of the pricing kernel $\boldsymbol{p}$ and the expectations kernel 1, utility is

$$u\left(\langle \boldsymbol{p},\boldsymbol{y}\rangle,\langle \mathbf{1},\boldsymbol{y}\rangle,\left\langle \boldsymbol{y},\left(\mathsf{I}-\mathbf{1}\mathbf{1}^{\top}\right)\boldsymbol{y}\right\rangle\right).$$

The matrix $\mathsf{I}-\mathbf{1}\mathbf{1}^{\top}$ is singular, since there exists a risk-free asset.

First-Order Necessary Condition

The first-order necessary condition for utility maximization is

$$\frac{\partial u}{\partial \boldsymbol{y}} = u_c \boldsymbol{p} + u_m \mathbf{1} + 2u_v \left(\mathsf{I} - \mathbf{1}\mathbf{1}^\top \right) \boldsymbol{y} = \mathbf{0}. \tag{1}$$

Optimum Saving

Premultiplying the first-order condition (1) by $\mathbf{1}^\top$ gives

$$u_c \langle \mathbf{1}, \boldsymbol{p} \rangle + u_m = 0,$$

which we interpret as the first-order condition for optimum saving. Investing one dollar in the risk-free asset lowers cost by one dollar and raises the mean by $1/\langle \mathbf{1}, \boldsymbol{p} \rangle$ dollars. Utility changes by

$$-u_c + u_m / \langle \mathbf{1}, \boldsymbol{p} \rangle,$$

which must be zero at the optimum.

Using the first-order condition for optimum saving, rearrange the first-order condition as (1) as

$$\left(\mathsf{I} - \mathbf{1}\mathbf{1}^\top\right)\boldsymbol{y} = -\frac{1}{2u_v}\left(u_c\boldsymbol{p} + u_m\mathbf{1}\right)$$
$$= -\frac{u_m}{2u_v}\left(\frac{u_c}{u_m}\boldsymbol{p} + \mathbf{1}\right)$$
$$= \frac{1}{a}\left(-\frac{1}{\langle \mathbf{1}, \boldsymbol{p}\rangle}\boldsymbol{p} + \mathbf{1}\right).$$

Solution Set

For there to be a solution to the rearranged first-order condition, necessarily $-\boldsymbol{p}/\langle\mathbf{1},\boldsymbol{p}\rangle+\mathbf{1}$ must lie in the range of $\mathsf{I}-\mathbf{1}\mathbf{1}^\top$. The solution set is then

$$\left\{\left(\mathsf{I}-\mathbf{1}\mathbf{1}^\top\right)^{+}\frac{1}{a}\left(-\frac{1}{\langle\mathbf{1},\boldsymbol{p}\rangle}\boldsymbol{p}+\mathbf{1}\right)\right\}+\mathrm{N}\left(\mathsf{I}-\mathbf{1}\mathbf{1}^\top\right).$$

The matrix $\mathsf{I} - \mathbf{1}\mathbf{1}^\top$ is singular and equals its generalized inverse.

$$\left(\mathsf{I} - \mathbf{1}\mathbf{1}^\top\right)^+ = \mathsf{I} - \mathbf{1}\mathbf{1}^\top.$$

The null space is any multiple of $\mathbf{1}$,

$$\mathrm{N}\left(\mathsf{I} - \mathbf{1}\mathbf{1}^\top\right) = \mathbf{R}\{\mathbf{1}\}.$$

The solution set is therefore

$$\boldsymbol{y} \in \left\{ \frac{1}{a} \left(- \frac{\left(\mathsf{I} - \mathbf{1}\mathbf{1}^\top\right)\boldsymbol{p}}{\langle \mathbf{1}, \boldsymbol{p} \rangle} \right) \right\} + \mathbf{R}\{\mathbf{1}\}.$$

Selecting the real number to give cost $c = \langle \boldsymbol{p}, \boldsymbol{y} \rangle$, the unique solution is

$$\begin{aligned} \boldsymbol{y} &= -\frac{1}{a} \frac{\left(\mathsf{I} - \mathbf{1}\mathbf{1}^\top\right)\boldsymbol{p}}{\langle \boldsymbol{p}, \mathbf{1} \rangle} + \left[\frac{c}{\langle \boldsymbol{p}, \mathbf{1} \rangle} + \frac{1}{a} \frac{\boldsymbol{p}^\top \left(\mathsf{I} - \mathbf{1}\mathbf{1}^\top\right)\boldsymbol{p}}{\langle \boldsymbol{p}, \mathbf{1} \rangle^2} \right] \mathbf{1} \\ &= -\frac{1}{a} \frac{\boldsymbol{p}}{\langle \boldsymbol{p}, \mathbf{1} \rangle} + \left(\frac{c}{\langle \boldsymbol{p}, \mathbf{1} \rangle} + \frac{1}{a} \frac{\boldsymbol{p}^\top \boldsymbol{p}}{\langle \boldsymbol{p}, \mathbf{1} \rangle^2} \right) \mathbf{1}. \end{aligned}$$

The mean

$$\langle \mathbf{1}, \boldsymbol{y} \rangle = \frac{c}{\langle \boldsymbol{p}, \mathbf{1} \rangle} + \frac{1}{a} \frac{\boldsymbol{p}^\top \left(\mathsf{I} - \mathbf{1}\mathbf{1}^\top \right) \boldsymbol{p}}{\langle \boldsymbol{p}, \mathbf{1} \rangle^2} = \frac{c}{\mathrm{E}\left(\boldsymbol{p}\right)} + \frac{1}{a} \frac{\mathrm{Var}\left(\boldsymbol{p}\right)}{\left[\mathrm{E}\left(\boldsymbol{p}\right)\right]^2}.$$

The variance

$$\boldsymbol{y}^\top \left(\mathsf{I} - \mathbf{1}\mathbf{1}^\top \right) \boldsymbol{y} = \frac{1}{a^2} \frac{\boldsymbol{p}^\top \left(\mathsf{I} - \mathbf{1}\mathbf{1}^\top \right) \boldsymbol{p}}{\langle \boldsymbol{p}, \mathbf{1} \rangle^2} = \frac{1}{a^2} \frac{\mathrm{Var}\left(\boldsymbol{p}\right)}{\left[\mathrm{E}\left(\boldsymbol{p}\right)\right]^2}.$$

As the absolute risk aversion a rises, the mean and the variance both fall.