Consumption

Consider a consumer with utility

$$\int_t^\infty v(c_\tau)\,\mathrm{e}^{-\rho(\tau-t)}\,\mathrm{d}\tau.$$

He acts to maximize expected utility. Utility is increasing in consumption, $v' > 0$, and concave, $v'' < 0$.

The utility from consumption at time t is $v(c_t)$, and the utility from consumption at time $t+\mathrm{d}t$ is

$$v\left(c_{t+\mathrm{d}t}\right)\mathrm{e}^{-\rho\,\mathrm{d}t} = v\left(c_{t+\mathrm{d}t}\right)\left(1-\rho\,\mathrm{d}t\right).$$

Applying the standard formula

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

gives

$$\begin{aligned} e^{-\rho\,dt} &= 1 + (-\rho\,dt) + \frac{(-\rho\,dt)^2}{2!} + \frac{(-\rho\,dt)^3}{3!} + \cdots \\ &= 1 - \rho\,dt, \end{aligned}$$

as the higher-order terms are zero.

Saving

Consider a consumer maximizing expected utility. At time t, he chooses his optimum consumption c_t and an optimum portfolio with return $\mathrm{d}a$. At time $t+\mathrm{d}t$ his optimum consumption is $c_{t+\mathrm{d}t}$.

Alternatively, he could have increased his saving by s. His current consumption would fall to $c_t - s$. His wealth at time $t+\mathrm{d}t$ would rise by $s\,(1+\mathrm{d}a)$, and he could choose to consume this extra wealth at $t+\mathrm{d}t$, raising consumption to $c_{t+\mathrm{d}t} + s\,(1+\mathrm{d}a)$.

As a function of s, expected utility during the two periods is

$$v(c_t - s) + \mathrm{E}_t \left\{ v\left[c_{t+\mathrm{d}t} + s\left(1 + \mathrm{d}a\right)\right]\left(1 - \rho\,\mathrm{d}t\right)\right\}.$$

The expectation is conditional on the information at time t.

First-Order Condition

The derivative of expected utility with respect to s is

$$-v'(c_t - s) + \mathrm{E}_t \left\{ v' [c_{t+\mathrm{d}t} + s(1 + \mathrm{d}a)] (1 + \mathrm{d}a)(1 - \rho\, \mathrm{d}t) \right\}.$$

A first-order condition for optimum behavior is that at $s = 0$ this derivative must be zero:

$$\begin{aligned} v'(c_t) &= \mathrm{E}_t \left[v'(c_{t+\mathrm{d}t})(1 + \mathrm{d}a)(1 - \rho\, \mathrm{d}t) \right] \\ &= \mathrm{E}_t \left[v'(c_{t+\mathrm{d}t})(1 + \mathrm{d}a - \rho\, \mathrm{d}t) \right] \qquad (1) \end{aligned}$$

Relationship of Consumption and Asset Return

The Itô formula for the change in the marginal utility of consumption is

$$
\begin{aligned}
v'(c_{t+\mathrm{d}t}) &= v'(c_t) + \mathrm{d}v' \\
&= v'(c_t) + v''(c_t)\,\mathrm{d}c_t + \frac{1}{2}v'''(c_t)\,(\mathrm{d}c_t)^2 .
\end{aligned}
$$

Substituting this formula into (1) gives

$$
\begin{aligned}
v'(c_t) &= \mathrm{E}_t\left[v'(c_{t+\mathrm{d}t})(1+\mathrm{d}a-\rho\,\mathrm{d}t)\right] \\
&= \mathrm{E}_t\left\{\left[v'(c_t)+v''(c_t)\,\mathrm{d}c_t+\frac{1}{2}v'''(c_t)(\mathrm{d}c_t)^2\right]\right. \\
&\qquad \left.(1+\mathrm{d}a-\rho\,\mathrm{d}t)\right\} \\
&= \mathrm{E}_t\left[v'(c_t)(1+\mathrm{d}a-\rho\,\mathrm{d}t)\right] \\
&\qquad +\mathrm{E}_t\left[v''(c_t)\,\mathrm{d}c_t(1+\mathrm{d}a)\right] \\
&\qquad +\mathrm{E}_t\left[\frac{1}{2}v'''(c_t)(\mathrm{d}c_t)^2\right].
\end{aligned}
$$

Interpretation

This equation shows the relationship between consumption and the asset return, under optimum choice.

It does not say anything about the direction of causality. Given a stochastic process for the asset return, one can figure out the implications for consumption. Or, given a stochastic process for consumption, one can figure out the implications for the asset return.

The relationship is also quite general. For example, it is flexible about how the individual pays for consumption—there might be stochastic labor income affecting both consumption and the portfolio choice.

No Uncertainty

If there were no uncertainty, then $\mathrm{d}a = r\,\mathrm{d}t$, in which r is the risk-free rate of return. The change in consumption would also have no instantaneous stochastic part.

Then the first-order condition

$$v'(c_t) = \mathrm{E}_t\left[v'(c_{t+\mathrm{d}t})(1 + r\,\mathrm{d}t - \rho\,\mathrm{d}t)\right]$$

becomes

$$\frac{v'(c_t)}{v'(c_{t+\mathrm{d}t})(1-\rho\,\mathrm{d}t)} = 1 + r\,\mathrm{d}t.$$

The left-hand side is the marginal rate of substitution between current and future consumption; the right-hand side is the relative price of current and future consumption. Consequently the first-order condition (1) is a generalization to uncertainty of this standard condition for an optimum.

The first-order condition

$$\begin{aligned} v'(c_t) &= \mathrm{E}_t\left[v'(c_t)\,(1+\mathrm{d}a-\rho\,\mathrm{d}t)\right] \\ &+\mathrm{E}_t\left[v''(c_t)\,\mathrm{d}c_t\,(1+\mathrm{d}a)\right] \\ &+\mathrm{E}_t\left[\frac{1}{2}v'''(c_t)\,(\mathrm{d}c_t)^2\right] \end{aligned}$$

becomes

$$v'(c_t) = v'(c_t)\,(1+\mathrm{d}a-\rho\,\mathrm{d}t)+v''(c_t)\,\mathrm{d}c_t\,.$$

Solving for the change in consumption gives

$$\begin{aligned} \mathrm{d}c_t &= \frac{v'(c_t)}{v''(c_t)}(\rho\,\mathrm{d}t - \mathrm{d}a) \\ &= \left[-\frac{v'(c_t)}{v''(c_t)}\right](r-\rho)\,\mathrm{d}t. \end{aligned}$$

As $-v'(c_t)/v''(c_t)$ is positive, consumption rises if the interest rate is greater than the rate of time preference, $r > \rho$.

Constant Relative Risk Aversion

Suppose that there is constant relative risk aversion of consumption,

$$v(c) = \begin{cases} \frac{c^{1-\alpha}}{1-\alpha} \text{ for } \alpha \neq 1 \\ \ln c \text{ for } \alpha = 1. \end{cases}$$

Then

$$v' = c^{-\alpha}$$

$$v'' = -\alpha c^{-\alpha-1} = -\alpha v'/c$$

$$v''' = \alpha(\alpha+1)c^{-\alpha-2} = \alpha(\alpha+1)v'/c^2.$$

Substituting into the first-order condition gives

$$
\begin{aligned}
v' &= \mathrm{E}_t\left[v'\left(1+\mathrm{d}a-\rho\,\mathrm{d}t\right)\right]+\mathrm{E}_t\left[v''\,\mathrm{d}c\left(1+\mathrm{d}a\right)\right]+\mathrm{E}_t\left[\frac{1}{2}v'''\left(\mathrm{d}c\right)^2\right]\\
&= \mathrm{E}_t\left[v'\left(1+\mathrm{d}a-\rho\,\mathrm{d}t\right)\right]+\mathrm{E}_t\left[-\alpha v'\left(\frac{\mathrm{d}c}{c}\right)\left(1+\mathrm{d}a\right)\right]\\
&\quad +\mathrm{E}_t\left[\frac{1}{2}\alpha\left(\alpha+1\right)v'\left(\frac{\mathrm{d}c}{c}\right)^2\right].
\end{aligned}
$$

Cancel out v' to get

$$0 = \mathrm{E}_t(\mathrm{d}a - \rho\,\mathrm{d}t) + \mathrm{E}_t\left[-\alpha\left(\frac{\mathrm{d}c}{c}\right)(1+\mathrm{d}a)\right] + \mathrm{E}_t\left[\frac{1}{2}\alpha(\alpha+1)\left(\frac{\mathrm{d}c}{c}\right)^2\right].$$

Let us look for a constant solution. Suppose that both consumption growth and the asset return have constant mean, variance, and covariance:

$$\frac{\mathrm{d}c}{c} \sim \mathrm{N}\left(\mu\,\mathrm{d}t, \sigma^2\,\mathrm{d}t\right)$$

$$\mathrm{d}a \sim \mathrm{N}\left(m\,\mathrm{d}t, s^2\,\mathrm{d}t\right),$$

correlation η. Substituting into the first-order condition gives

$$0 = \left[(m-\rho)\,\mathrm{d}t\right] + \left[-\alpha\,(\mu+\eta s\sigma)\,\mathrm{d}t\right] + \left[\frac{1}{2}\alpha\,(\alpha+1)\,\sigma^2\,\mathrm{d}t\right],$$

and one can cancel out $\mathrm{d}t$. However the relationship is complex.