Expected Utility

Let $u(w_t)$ denote expected lifetime utility at time t. The value is the optimum expected utility, derived from expectations about the future and optimizing behavior. Expected utility might be state dependent, and we would then write $u(w_t, s_t)$. For example, the expectation of future returns on assets or of future labor income would influence u.

First-Order Condition

Let us use the first-order condition for optimum portfolio choice:

$$0 = \mathrm{E}\left[u'\left(w_{t+\mathrm{d}t}\right)\left(\mathrm{d}a_i - \mathrm{d}a\right)\right].$$

Here $u'\left(w_{t+\mathrm{d}t}\right)$ is the marginal utility of end-of-period wealth, $\mathrm{d}a$ is the return on the optimum portfolio, and $\mathrm{d}a_i$ is the return on asset i.

Consumption

Consider a consumer with utility

$$\int_t^\infty v(c_\tau)\,\mathrm{e}^{-\rho(\tau-t)}\,\mathrm{d}\tau.$$

He acts to maximize expected utility.

The utility from consumption at time t is $v(c_t)$. Utility is increasing in consumption, $v' > 0$, and concave, $v'' < 0$.

First-Order Necessary Condition for Consumption

Optimizing behavior implies that the marginal utility of consumption equals the marginal expected utility of wealth,

$$v'(c_t) = u'(w_t).$$

For example, if the marginal utility of consumption were greater than the marginal expected utility of wealth, then consumption should be increased.

From this point of view, the marginal expected utility of wealth is also the marginal expected utility of saving.

Consequently a first-order necessary condition for optimum consumption and portfolio choice is

$$0 = \mathrm{E}\left[v'\left(c_{t+\mathrm{d}t}\right)\left(\mathrm{d}a_i - \mathrm{d}a\right)\right]. \tag{1}$$

We have just substituted the marginal utility of consumption for the marginal utility of wealth.

The Itô formula for the change in the marginal utility of consumption is

$$
\begin{aligned}
v'(c_{t+\mathrm{d}t}) &= v'(c_t) + \mathrm{d}v' \\
&= v'(c_t) + v''(c_t)\,\mathrm{d}c_t + \frac{1}{2}v'''(c_t)\,(\mathrm{d}c_t)^2 .
\end{aligned}
$$

Substituting this equation into the first-order condition (1) gives

$$
\begin{aligned}
0 &= \mathrm{E}\left[v'\left(c_{t+\mathrm{d}t}\right)\left(\mathrm{d}a_i - \mathrm{d}a\right)\right] \\
&= \mathrm{E}\left\{\left[v'\left(c_t\right) + v''\left(c_t\right)\mathrm{d}c_t + \frac{1}{2}v'''\left(c_t\right)\left(\mathrm{d}c_t\right)^2\right]\left(\mathrm{d}a_i - \mathrm{d}a\right)\right\} \\
&= \mathrm{E}\left\{\left[v'\left(c_t\right) + v''\left(c_t\right)\mathrm{d}c_t\right]\left(\mathrm{d}a_i - \mathrm{d}a\right)\right\} \\
&= v'\left(c_t\right)\mathrm{E}\left(\left\{1 - \left[-\frac{c_t v''\left(c_t\right)}{v'\left(c_t\right)}\right]\left(\frac{\mathrm{d}c_t}{c_t}\right)\right\}\left(\mathrm{d}a_i - \mathrm{d}a\right)\right).
\end{aligned}
$$

Define α as the relative risk aversion with respect to consumption,

$$\alpha := -\frac{c_t v''(c_t)}{v'(c_t)}.$$

(Note: This value can differ from the relative risk aversion with respect to wealth, $-wu''/u'$.)

Then the first-order condition says

$$0 = \mathrm{E}\left\{\left[1 - \alpha\left(\frac{\mathrm{d}c_t}{c_t}\right)\right](\mathrm{d}a_i - \mathrm{d}a)\right\}. \tag{2}$$

This first-order condition relates consumption growth $\mathrm{d}c/c$ to asset returns.

Consumption-Based CAPM

As equation (2) holds for any asset i, we could subtract the equation for one asset from the equation for another asset to get

$$0 = \mathrm{E}\left\{\left[1 - \alpha\left(\frac{\mathrm{d}c_t}{c_t}\right)\right](\mathrm{d}a_i - \mathrm{d}a_j)\right\}.$$

Take asset j as the risk-free asset, with return $\mathrm{d}a_j = r\,\mathrm{d}t$.

Then

$$
\begin{aligned}
\mathrm{E}\left(\mathrm{d}a_i\right) &= r\,\mathrm{d}t + \alpha\,\mathrm{d}a_i\left(\frac{\mathrm{d}c_t}{c_t}\right) \\
&= r\,\mathrm{d}t + \alpha\,\mathrm{Cov}\left[\mathrm{d}a_i, \left(\frac{\mathrm{d}c_t}{c_t}\right)\right].
\end{aligned}
$$

Thus the covariance of the asset return with consumption growth sets the risk premium.

Market Portfolio Versus Consumption

Which model seems better? Does covariance with the return on the market portfolio determine the risk premium on an asset, or does covariance with consumption growth determine it?

If the marginal utility of expected wealth is not state dependent, then both models are equivalent.

The key advantage of the consumption-based model is that consumption is readily measured. In contrast, it is difficult to measure total wealth. In particular, what is the value of human capital? The consumption-based model need not deal with this question.

A weakness of the consumption-based model is its dependence on a particular utility function: utility is separable intertemporally.

Correlation of Asset Returns

In the stock market, stock prices do tend to rise and fall together. The return on a typical stock is strongly correlated with the stock market indices, such as the Standard and Poor's 500 stock price index. This correlation means that potentially the market portfolio-based capital-asset pricing model might explain risk premiums.

However the correlation between stock and bond returns is near zero. In contrast, economic theory predicts a strong correlation.

Overall, stock returns are strongly correlated with each other, but not with anything else. In contrast, economic theory predicts a correlation with interest rates, the business cycle, etc.

Low Correlation of Consumption Growth with Asset Returns

A major failing of the consumption-based capital-asset pricing model is that asset returns have little correlation with consumption growth. This finding holds for both stocks and bonds.

One could regard this low correlation as a problem for finance theory.

However it is perhaps a bigger problem for macroeconomic theory. According to the standard theory of the consumer, high asset returns raise wealth and thus should increase consumption significantly. Yet large fluctuations of stock prices have little discernable affect on consumption.