# Capital-Asset Pricing Model

Sharpe [1] presents the capital-asset pricing model, a theory of the risk premium on a capital asset in market equilibrium.

**Definition 1 (Risk Premium)** *The risk premium on an asset is its expected rate of return less the rate of return on a risk-free asset.*

Equivalently, the risk premium is the expected excess return.

We present Sharpe's reasoning, while working in a small-risk context.

## Homogeneous Expectations

We assume homogeneous expectations. Everyone has the same probability distribution for rates of return.

Of course this assumption is a limitation, as investors do not agree about expected returns.

# Risk Premium?

In the simplest financial model, the risk premium is zero: the expected rate of return is the same for all assets, the "market interest rate."

As people are risk averse, however, it is natural that a risky asset should have a positive risk premium. If the risk premium were zero rather than positive, why would anyone buy the asset?

But what makes an asset risky? And what determines the magnitude of the risk premium?

If there is high demand for an asset, then the asset price will be high, and consequently the expected rate of return will be low. What determines the demand for an asset?

# Variance?

When risks are small, expected utility is the expected rate of return less half the relative risk aversion times the variance of the rate of return,

$$\mathrm{E}\,(\mathrm{d}a) - \frac{1}{2}\alpha \mathrm{Var}\,(\mathrm{d}a)\,.$$

Since the expected utility involves a tradeoff of mean and variance, one might conjecture that the risk premium is proportional to the variance.

However the capital-asset pricing model says that this conjecture is false.

# Diversified Portfolio

To reduce risk, a risk-averse investor does not invest solely in a single asset but instead buys a diversified portfolio.

The capital-asset pricing model says that the demand for an asset depends on how the usefulness of the asset for diversification, as well as on its expected rate of return.

# Negative Correlation

Consider an asset for which the rate of return is negatively correlated with the rate of return on other assets. This asset is unusually good for diversification, so there is high demand for the asset. The asset price should be high, and the expected rate of return should be low.

# High Correlation

Conversely, consider an asset for which the rate of return is highly correlated with the rate of return on other assets. This asset is not useful for diversification, so the demand for the asset is low. The asset price should be low, and the expected rate of return should be high.

# Low Transactions Costs

The capital-asset pricing model assumes that transactions costs are low (zero), so diversification is not costly.

# Supply: Market Portfolio

**Definition 2 (Market Portfolio)** *The market portfolio is the outstanding stock of all assets in the economy.*

One can subdivide the market portfolio into the risky assets and the risk-free asset. The *market portfolio of risky assets* is the outstanding stock of all risky assets in the economy.

# Demand: Separation Theorem

The separation theorem says that the optimum portfolio choice is to invest partly in the efficient portfolio of risky assets and partly in the risk-free asset. The risk aversion determines the proportion of wealth invested in each.

Consequently, the aggregate market demand is to invest partly in the efficient portfolio of risky assets and partly in the risk-free asset.

# Market Equilibrium

Market equilibrium requires that demand equals supply. The supply is the market portfolio, so in equilibrium demand must equal the market portfolio.

# Separation Theorem

In the separation theorem, the difference between investors is the risk aversion. An investor with higher relative risk aversion buys more of the risk-free asset and less of the efficient portfolio of risky assets.

Thus the following two conditions are necessary and sufficient for market equilibrium:

- the efficient portfolio of risky assets must equal the market portfolio of risky assets;
- the fraction of wealth invested in the risky assets versus the risk-free asset must agree with supply.

## Market Equilibrium

As the market portfolio is a combination of the market portfolio of risky assets and the risk-free asset, we can restate these necessary and sufficient conditions for market equilibrium in terms of the market portfolio.

**Theorem 3 (Market Equilibrium)** *The following two conditions are necessary and sufficient for market equilibrium:*

- *the market portfolio is efficient;*
- *along the efficient frontier, the typical investor chooses the market portfolio.*

Of course the second condition cannot hold without the first, but the first does not imply the second.

# Two-Asset Portfolio Choice

For an individual investing partly in a risky asset and partly in a risk-free asset, the fraction of wealth invested in the risky asset is

$$f = \frac{m}{\alpha s^2}.$$

Here $m$ is the risk premium on the risky asset, $s$ is the standard deviation of is rate of return, and $\alpha$ is the relative risk aversion.

Apply this model to equilibrium: the risky asset is the market portfolio. Market equilibrium requires $f = 1$.

**Theorem 4 (Risk Premium on the Market Portfolio)** *In market equilibrium, the risk premium on the market portfolio is the relative risk aversion of a typical investor multiplied by the variance of the return on the market portfolio,*

$$\mathrm{E}\left(\mathrm{d}x_m\right) = \alpha \operatorname{Var}\left(\mathrm{d}x_m\right). \tag{1}$$

Here $\mathrm{d}x_m$ is the excess return on the market portfolio.

Thus the conjecture that variance sets the risk premium does hold for the market portfolio. However it does *not* hold for an individual asset.

# Least-Squares Linear Regression

Consider the least-squares linear regression of an excess return $\mathrm{d}x$ on the excess return $\mathrm{d}x_m$ on the market portfolio,

$$\mathrm{d}x = \gamma \mathrm{dt} + \beta \, \mathrm{d}x_m + \mathrm{d}z. \tag{2}$$

The error has expected value zero: $\mathrm{E}(\mathrm{d}z) = 0$. Its variance $(\mathrm{d}z)^2$ need not be one, but it is uncorrelated with the excess return on the market portfolio: $\mathrm{d}z\,\mathrm{d}x_m = 0$.

Also,

$$
\begin{aligned}
\frac{\mathrm{Cov}(\mathrm{d}x, \mathrm{d}x_m)}{\mathrm{Var}(\mathrm{d}x_m)} &= \frac{\mathrm{d}x\,\mathrm{d}x_m}{(\mathrm{d}x_m)^2} \\
&= \frac{(\gamma \mathrm{dt} + \beta\,\mathrm{d}x_m + \mathrm{d}z)\,\mathrm{d}x_m}{(\mathrm{d}x_m)^2} \\
&= \beta.
\end{aligned}
$$

# Beta Coefficient

The beta coefficient $\beta$ summarizes the relationship of the rate of return to the rate of return on the market portfolio. For a typical asset, necessarily $\beta = 1$; an extra 10% return on the market portfolio means that a typical asset has an extra 10% return. An asset for which the return is negatively correlated with other assets has $\beta < 0$.

Consider a portfolio in which one invests the fraction $f$ of wealth in a particular asset and the fraction $1-f$ in the market portfolio. The excess return is

$$f\,\mathrm{d}x + (1-f)\,\mathrm{d}x_m = f\,(\gamma\,\mathrm{dt} + \beta\,\mathrm{d}x_m + \mathrm{d}z) + (1-f)\,\mathrm{d}x_m.$$

The expected excess return is

$$\begin{aligned}
&\mathrm{E}\left[f\,\mathrm{d}x + (1-f)\,\mathrm{d}x_m\right] \\
&\quad = \mathrm{E}\left[f\,(\gamma\mathrm{dt} + \beta\,\mathrm{d}x_m + \mathrm{d}z) + (1-f)\,\mathrm{d}x_m\right] \\
&\quad = f\gamma\mathrm{dt} + \left[f\beta + (1-f)\right]\mathrm{E}\left(\mathrm{d}x_m\right) + f\mathrm{E}\left(\mathrm{d}z\right) \\
&\quad = f\gamma\mathrm{dt} + \left[f\beta + (1-f)\right]\mathrm{E}\left(\mathrm{d}x_m\right),
\end{aligned}$$

since $\mathrm{E}\left(\mathrm{d}z\right) = 0$.

The variance is

$$\begin{aligned}
&\operatorname{Var}\left[f\,\mathrm{d}x + (1-f)\,\mathrm{d}x_m\right] \\
&\quad = \operatorname{Var}\left[f\,(\gamma\,\mathrm{dt} + \beta\,\mathrm{d}x_m + \mathrm{d}z) + (1-f)\,\mathrm{d}x_m\right] \\
&\quad = \operatorname{Var}\left[\left[f\beta + (1-f)\right]\mathrm{d}x_m + f\,\mathrm{d}z\right] \\
&\quad = \left[f\beta + (1-f)\right]^2(\mathrm{d}x_m)^2 + f^2\,(\mathrm{d}z)^2\,,
\end{aligned}$$

since $\mathrm{d}x_m\,\mathrm{d}z = 0$, by the theory of least-squares linear regression.

As $f$ changes, one can trace out how the mean and the variance change. The change in the expected value is

$$\frac{\mathrm{dE}(\cdot)}{\mathrm{d}f} = \gamma \mathrm{dt} + (\beta - 1)\,\mathrm{E}(\mathrm{d}x_m).$$

The change in the standard deviation is

$$
\begin{aligned}
\frac{\mathrm{dSd}(\cdot)}{\mathrm{d}f} &= \frac{\mathrm{dSd}}{\mathrm{dVar}} \frac{\mathrm{dVar}(\cdot)}{\mathrm{d}f} \\
&= \frac{1}{\left(\frac{\mathrm{dVar}}{\mathrm{dSd}}\right)} \frac{\mathrm{dVar}(\cdot)}{\mathrm{d}f} \\
&= \frac{1}{2\,\mathrm{Sd}} \left[2\left[f\beta + (1-f)\right](\beta - 1)(\mathrm{d}x_m)^2 + 2f(\mathrm{d}z)^2\right] \\
&= \frac{1}{\mathrm{Sd}(\mathrm{d}x_m)} (\beta - 1)\,\mathrm{Var}(\mathrm{d}x_m) \ \text{ at } f = 0 \\
&= (\beta - 1)\,\mathrm{Sd}(\mathrm{d}x_m)\,.
\end{aligned}
$$

The slope of the parametric curve is

$$\frac{\frac{\mathrm{dE}(\cdot)}{\mathrm{d}f}}{\frac{\mathrm{dSd}(\cdot)}{\mathrm{d}f}} = \frac{\gamma \mathrm{d}t + (\beta - 1)\,\mathrm{E}\,(\mathrm{d}x_m)}{(\beta - 1)\,\mathrm{Sd}\,(\mathrm{d}x_m)},$$

at $f = 0$.

The slope of the efficient frontier is the expected excess return on the market portfolio divided by the standard deviation of the return,

$$\frac{\mathrm{E}\,(\mathrm{d}x_m)}{\mathrm{Sd}\,(\mathrm{d}x_m)}.$$

# Tangent

The parametric curve must be tangent to the efficient frontier at $f = 0$. Otherwise there would be a contradiction: the curve would cross the efficient frontier, which would mean that the frontier is not efficient.

The slopes are equal if and only if $\gamma = 0$.

# Risk Premium

**Theorem 5 (Risk Premium)** *The risk premium is proportional to the beta coefficient,*

$$\mathrm{E}\left(\mathrm{d}x\right) = \beta\,\mathrm{E}\left(\mathrm{d}x_m\right).$$

That this condition holds for any portfolio is necessary and sufficient for the mean and standard deviation of the market portfolio to lie on the efficient frontier

In this case the first condition in theorem 3 holds. If the market-portfolio risk-premium condition (1) also holds, then the second condition in theorem 3 holds as well.

# Systematic and Non-Systematic Risk

**Definition 6** *In the least-squares linear regression*

$$\mathrm{d}x = \gamma\,\mathrm{dt} + \beta\,\mathrm{d}x_m + \mathrm{d}z,$$

*the term*

$$\beta\,\mathrm{d}x_m$$

*is the systematic risk, and the term*

$$\mathrm{d}z$$

*is the non-systematic risk.*

# Intuitive Interpretation

The capital-asset pricing model says that the systematic risk determines the risk premium, and the non-systematic risk has no effect on the risk premium.

The intuitive explanation is that the non-systematic risk can be eliminated by diversification.

In contrast, the systematic risk is aggregate, economy-wide risk that must be borne by someone. It cannot be eliminated by diversification, so whoever bears this risk is compensated for doing so, by the risk premium.

## Econometric Test

A simple econometric test for whether the market-portfolio is efficient is whether the intercept in the least-squares linear regression (2) is zero.

# Stocks, not Flows

The portfolio demand and supply deals with stocks, not flows.

The outstanding stock of assets influences the risk premium, by affecting the beta coefficient.

# References

[1] William F. Sharpe. Capital asset prices: A theory of market equilibrium under conditions of risk. *Journal of Finance*, XIX(3):425–442, September 1964. HG1J6. 1