Probability Density Function for Wiener-Brownian Motion

Let \( p(x,t) \) denote the probability density function for \( x \) at time \( t \). For Wiener-Brownian motion,

\[
p(x,t) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t},
\]
as \( x \) has mean zero and variance \( t \).

At time zero, the probability is the Dirac delta function

\[ p(x,0) = \delta(x). \]

All probability is concentrated at zero: by definition

\[ \int_{x} \delta(x) \, dx = 1, \text{ but } \delta(x) = 0 \text{ for } x \neq 0. \]

Forward Equation

The *forward equation* describes how the probability density function evolves as time passes, starting from an arbitrary initial probability density \( p(x,0) \).

Heat Equation

For Wiener-Brownian motion, differentiation of the probability density function shows that it satisfies

\[
p_t = \frac{1}{2} p_{xx}.
\]

In physics, this *heat equation* describes the diffusion of heat: \( p \) is the distribution of heat in space, and the equation shows how it diffuses as time passes.

Binomial Random Walk

We show that Wiener-Brownian motion is the limit of a binomial random walk, by analyzing the forward equation. Consider a discrete-time, binomial random walk, for which \( x \) either rises or falls each period. Initially \( x = 0 \). Let the period length be \( \Delta t/2 \). The random variable \( x \) rises or falls by \( \Delta x/2 \), with equal probability. By looking at even periods only, we can work with a fixed grid of \( x \) values \( \ldots, -\Delta x, 0, \Delta x, \ldots \) and times \( 0, \Delta t, 2\Delta t, \ldots \).

Over two periods, \( x \) rises by \( \Delta x \) with probability \( 1/4 \), stays constant with probability \( 1/2 \), and falls by \( \Delta x \) with probability \( 1/4 \). After two periods,

\[
\text{Var}(x_{2\Delta t}) = \frac{1}{4} (\Delta x)^2 + \frac{1}{2} (0)^2 + \frac{1}{4} (\Delta x)^2 = \frac{1}{2} (\Delta x)^2.
\]

After \( 2n \) periods (time \( t = 2n \times \Delta t/2 = n\Delta t \)),

\[
\text{Var}(x_t) = \frac{1}{2} n (\Delta x)^2.
\]
For Wiener-Brownian motion, this variance is \( t \), so we require
\[
t = n\Delta t = \frac{1}{2}n(\Delta x)^2.
\]
Maintaining the relationship
\[
\Delta t = \frac{1}{2}(\Delta x)^2,
\]
we take the limit as \( \Delta t \to 0 \) and \( n \to \infty \), such that \( t = n\Delta t \).

Hence
\[
p(x, t + \Delta t) - p(x, t) = \frac{1}{4}p(x + \Delta x, t) - \frac{1}{2}p(x, t) + \frac{1}{4}p(x - \Delta x, t)
\]
\[
= \frac{1}{4}[p(x + \Delta x, t) - p(x, t)]
\]
\[
- \frac{1}{4}[p(x, t) - p(x - \Delta x, t)].
\]

**Forward Equation**

Let \( p(x, t) \) denote the probability density function for \( x \) at time \( t \). The initial condition says \( p(0, 0) = 1/\Delta x \) and \( p(x, 0) = 0 \) for \( x \neq 0 \) (the discrete analogue of the Dirac delta function).

The forward equation is a discrete approximation to the heat equation. The forward equation is
\[
p(x, t + \Delta t) = \frac{1}{4}p(x + \Delta x, t) + \frac{1}{2}p(x, t) + \frac{1}{4}p(x - \Delta x, t).
\]

Dividing by \( \Delta t = \frac{1}{2}(\Delta x)^2 \) gives
\[
\frac{p(x, t + \Delta t) - p(x, t)}{\Delta t} = \frac{1}{\Delta x} \frac{p(x + \Delta x) - p(x)}{\Delta x} = \frac{p(x) - p(x - \Delta x)}{\Delta x}.
\]

Taking the limit yields the heat equation.

**Excellent Approximation**

It follows that the binomial can approximate Wiener-Brownian motion arbitrarily well, and in this sense we have shown that Wiener-Brownian motion is a well-defined stochastic process.

In fact the approximation is excellent. The following table shows the excellent quality of the approximation, for \( t = 1 \), \( n = 8 \), \( \Delta t = 1/8 \), \( \Delta x = .5 \). The values for the binomial are remarkably close to the values for the unit normal in the final column.