Probability Density Function for Wiener-Brownian Motion

Let $p(x,t)$ denote the probability density function for x at time t. For Wiener-Brownian motion,

$$p(x,t) = \frac{1}{\sqrt{2\pi t}} \mathrm{e}^{-x^2/2t},$$

as x has mean zero and variance t.

At time zero, the probability is the Dirac delta function

$$p(x,0) = \delta(x).$$

All probability is concentrated at zero: by definition $\int_x \delta(x)\,\mathrm{d}x = 1$, but $\delta(x) = 0$ for $x \neq 0$.

Forward Equation

The *forward equation* describes how the probability density function evolves as time passes, starting from an arbitrary initial probability density $p(x,0)$.

Heat Equation

For Wiener-Brownian motion, differentiation of the probability density function shows that it satisfies

$$p_t = \frac{1}{2} p_{xx}.$$

In physics, this *heat equation* describes the diffusion of heat: p is the distribution of heat in space, and the equation shows how it diffuses as time passes.

Binomial Random Walk

We show that Wiener-Brownian motion is the limit of a binomial random walk, by analyzing the forward equation. Consider a discrete-time, binomial random walk, for which x either rises or falls each period. Initially $x = 0$. Let the period length be $\Delta t/2$. The random variable x rises or falls by $\Delta x/2$, with equal probability. By looking at even periods only, we can work with a fixed grid of x values $\ldots, -\Delta x, 0, \Delta x, \ldots$ and times $0, \Delta t, 2\Delta t, \ldots$.

Over two periods, x rises by Δx with probability $1/4$, stays constant with probability $1/2$, and falls by Δx with probability $1/4$. After two periods,

$$\mathrm{Var}(x_{\Delta t}) = \frac{1}{4}(\Delta x)^2 + \frac{1}{2}(0)^2 + \frac{1}{4}(\Delta x)^2 = \frac{1}{2}(\Delta x)^2 .$$

After $2n$ periods (time $t = 2n \times \Delta t/2 = n\Delta t$),

$$\mathrm{Var}(x_t) = \frac{1}{2} n (\Delta x)^2 .$$

For Wiener-Brownian motion, this variance is t, so we require $t = n\Delta t = \frac{1}{2}n(\Delta x)^2$. Maintaining the relationship

$$\Delta t = \frac{1}{2}(\Delta x)^2,$$

we take the limit as $\Delta t \to 0$ and $n \to \infty$, such that $t = n\Delta t$.

Forward Equation

Let $p(x,t)$ denote the probability density function for x at time t. The initial condition says $p(0,0) = 1/\Delta x$ and $p(x,0) = 0$ for $x \neq 0$ (the discrete analogue of the Dirac delta function).

The forward equation is a discrete approximation to the heat equation. The forward equation is

$$p(x, t + \Delta t)$$
$$= \frac{1}{4} p(x + \Delta x, t) + \frac{1}{2} p(x, t) + \frac{1}{4} p(x - \Delta x, t).$$

Hence

$$
\begin{aligned}
& p(x,t+\Delta t) - p(x,t) \\
&\quad = \frac{1}{4} p(x+\Delta x,t) - \frac{1}{2} p(x,t) + \frac{1}{4} p(x-\Delta x,t) \\
&\quad = \frac{1}{4} \left[p(x+\Delta x,t) - p(x,t) \right] \\
&\qquad - \frac{1}{4} \left[p(x,t) - p(x-\Delta x,t) \right].
\end{aligned}
$$

Dividing by $\Delta t = \frac{1}{2}(\Delta x)^2$ gives

$$\frac{p(x,t+\Delta t) - p(x,t)}{\Delta t} = \frac{1}{2}\frac{\frac{p(x+\Delta x,t)-p(x,t)}{\Delta x} - \frac{p(x,t)-p(x-\Delta x,t)}{\Delta x}}{\Delta x}.$$

Taking the limit yields the heat equation.

Excellent Approximation

It follows that the binomial can approximate Wiener-Brownian motion arbitrarily well, and in this sense we have shown that Wiener-Brownian motion is a well-defined stochastic process.

In fact the approximation is excellent. The following table shows the excellent quality of the approximation, for $t = 1$, $n = 8$, $\Delta t = 1/8$, $\Delta x = .5$. The values for the binomial are remarkably close to the values for the unit normal in the final column.

$t=0$	$t=1/8$	$t=1/4$	$t=3/8$	$t=1/2$	$t=5/8$	$t=3/4$	$t=7/8$	$t=1$	Normal	x
0	0	0	0	0	0	0	0	0.000	0.000	-4.0
0	0	0	0	0	0	0	0.000	0.000	0.001	-3.5
0	0	0	0	0	0	0.000	0.002	0.004	0.004	-3.0
0	0	0	0	0	0.002	0.006	0.011	0.017	0.018	-2.5
0	0	0	0	0.008	0.020	0.032	0.044	0.056	0.054	-2.0
0	0	0	0.031	0.063	0.088	0.107	0.122	0.133	0.130	-1.5
0	0	0.125	0.188	0.219	0.234	0.242	0.244	0.244	0.242	-1.0
0	0.500	0.500	0.469	0.438	0.410	0.387	0.367	0.349	0.352	-0.5
2.000	1.000	0.750	0.625	0.547	0.492	0.451	0.419	0.393	0.399	0.0
0	0.500	0.500	0.469	0.438	0.410	0.387	0.367	0.349	0.352	0.5
0	0	0.125	0.188	0.219	0.234	0.242	0.244	0.244	0.242	1.0
0	0	0	0.031	0.063	0.088	0.107	0.122	0.133	0.130	1.5
0	0	0	0	0.008	0.020	0.032	0.044	0.056	0.054	2.0
0	0	0	0	0	0.002	0.006	0.011	0.017	0.018	2.5
0	0	0	0	0	0	0.000	0.002	0.004	0.004	3.0
0	0	0	0	0	0	0	0.000	0.000	0.001	3.5
0	0	0	0	0	0	0	0	0.000	0.000	4.0