Random Walk

According to the random-walk theory, a stock price follows a random walk. The change in the stock price is white noise.

In contrast, the average stock price during a period of time does not follow a random walk. Its change from one period to the next is a first-order moving average, with first-order autocorrelation $\frac{1}{4}$.

In the past, some econometric studies of the average stock price have wrongly claimed to have disproved the random walk theory,

Stock Price

Suppose that the stock price z_t is Wiener-Brownian motion. The first-difference

$$\Delta z_t := z_{t+1} - z_t = \int_t^{t+1} \mathrm{d}z_\tau.$$

Its mean is zero.

Moving Average

Below we express variables as an integral (moving average) of the stock price changes $\mathrm{d}z_t$. We then calculate a variance or covariance using the relation

$$\mathrm{E}\left(\mathrm{d}z_t\, \mathrm{d}z_\tau\right) = \begin{cases} \mathrm{d}t \text{ if } t = \tau \\ 0 \text{ if } t \neq \tau. \end{cases}$$

The variance of the first difference

$$\mathrm{Var}\left(\Delta z_t\right) = \mathrm{E}\left[\left(\int_t^{t+1} \mathrm{d}z_\tau\right)^2\right] = \int_t^{t+1} \mathrm{E}\left[\left(\mathrm{d}z_\tau\right)^2\right] = \int_t^{t+1} \mathrm{d}\tau = 1.$$

The first-order autocorrelation of Δz_t is zero, and the first-difference is white noise.

Average Stock Price

Let y_t denote the average stock price from time t to time $t+1$:

$$\begin{aligned} y_t &:= \int_t^{t+1} z_\tau\, \mathrm{d}\tau \\ &= \int_t^{t+1} \left(z_t + \int_t^{\tau} \mathrm{d}z_w\right) \mathrm{d}\tau \\ &= z_t + \int_t^{t+1} \left(\int_w^{t+1} \mathrm{d}\tau\right) \mathrm{d}z_w \\ &= z_t + \int_t^{t+1} (t+1-w)\, \mathrm{d}z_w. \end{aligned}$$

The average stock price is not a random walk.

Its first difference is

$$\begin{aligned} \Delta y_t &:= y_{t+1} - y_t \\ &= \left[z_{t+1} + \int_{t+1}^{t+2} (t+2-w)\, \mathrm{d}z_w\right] \\ &\quad - \left[z_t + \int_t^{t+1} (t+1-w)\, \mathrm{d}z_w\right] \\ &= \int_t^{t+1} \mathrm{d}z_\tau + \int_{t+1}^{t+2} (t+2-\tau)\, \mathrm{d}z_\tau + \int_t^{t+1} (\tau - t - 1)\, \mathrm{d}z_\tau \\ &= \int_{t+1}^{t+2} (t+2-\tau)\, \mathrm{d}z_\tau + \int_t^{t+1} (\tau - t)\, \mathrm{d}z_\tau. \end{aligned}$$

Weighted Average

The integration expresses the first difference as a weighted average of the stock price changes dz_τ, with all weights positive. The sum of the coefficients is one:

$$\begin{aligned}\int_{t+1}^{t+2} (t+2-\tau)\,d\tau + \int_{t}^{t+1} (\tau - t)\,d\tau \\ &= \int_0^1 w\, dw + \int_0^1 w\, dw \\ &= \frac{1}{2} + \frac{1}{2} \\ &= 1.\end{aligned}$$

The variance

$$\begin{aligned}\mathrm{Var}(\Delta y_t) &= \mathrm{E}\left[(\Delta y_t)^2\right] \\ &= \int_{t+1}^{t+2} (t+2-\tau)^2\,d\tau + \int_{t}^{t+1} (t-\tau)^2\,d\tau \\ &= 2\int_0^1 \tau^2\,d\tau \\ &= \frac{2}{3},\end{aligned}$$

The change in the average stock price has a smaller variance than the change in the stock price.

The first-order covariance

$$\begin{aligned}&\mathrm{Cov}(\Delta y_{t+1}, \Delta y_t) \\ &\quad= \mathrm{E}\left[\left(-\int_{t+1}^{t+2} (t+1-\tau)\,dz_\tau\right)\left(\int_{t+1}^{t+2} (t+2-\tau)\,dz_\tau\right)\right] \\ &\quad= -\int_{t+1}^{t+2} (t+1-\tau)(t+2-\tau)\,d\tau \\ &\quad= \int_0^1 w(1-w)\,dw \text{ (substituting } w := \tau-(t+1)) \\ &\quad= \left(\frac{1}{2}w^2 - \frac{1}{3}w^3\right)\Big]_0^1 \\ &\quad= \frac{1}{6}.\end{aligned}$$

First-Order Moving Average

The average stock price y_t is not a random walk. Its change is not white noise, but rather is a first-order moving average, with first-order autocorrelation $\frac{1}{4}$.