Random Walk

According to the random-walk theory, a stock price follows a random walk. The change in the stock price is white noise. In contrast, the average stock price during a period of time does not follow a random walk. Its change from one period to the next is a first-order moving average, with first-order autocorrelation $\frac{1}{4}$.

In the past, some econometric studies of the average stock price have wrongly claimed to have disproved the random walk theory,

Stock Price

Suppose that the stock price $z_t$ is Wiener-Brownian motion. The first-difference

$$\Delta z_t := z_{t+1} - z_t = \int_t^{t+1} dz_\tau.$$  

Its mean is zero.

Moving Average

Below we express variables as an integral (moving average) of the stock price changes $dz_t$. We then calculate a variance or covariance using the relation

$$E(dz_t dz_\tau) = \begin{cases} 
  dt & \text{if } t = \tau \\
  0 & \text{if } t \neq \tau.
\end{cases}$$

The variance of the first difference

$$\text{Var}(\Delta z_t) = E\left[\left(\int_t^{t+1} dz_\tau\right)^2\right] = \int_t^{t+1} E\left[(dz_\tau)^2\right] = \int_t^{t+1} d\tau = 1.$$  

The first-order autocorrelation of $\Delta z_t$ is zero, and the first-difference is white noise.

Average Stock Price

Let $y_t$ denote the average stock price from time $t$ to time $t+1:

$$y_t := \int_t^{t+1} z_\tau d\tau = \int_t^{t+1} z_\tau d\tau + \int_t^{t+1} (t+1-w) dw$$

The average stock price is not a random walk.

Its first difference is

$$\Delta y_t := y_{t+1} - y_t$$

$$= \left[z_{t+1} + \int_{t+1}^{t+2} (t+2-w) dw\right]$$

$$- \left[z_t + \int_t^{t+1} (t+1-w) dw\right]$$

$$= \int_t^{t+1} dz_\tau + \int_{t+1}^{t+2} (t+2-\tau) dz_\tau + \int_t^{t+1} (\tau - 1) dz_\tau$$

$$= \int_{t+1}^{t+2} (t+2-\tau) dz_\tau + \int_t^{t+1} (\tau - t) dz_\tau.$$
### Weighted Average

The integration expresses the first difference as a weighted average of the stock price changes \(dz_\tau\), with all weights positive. The sum of the coefficients is one:

\[
\int_{t+1}^{t+2} (t+2-\tau) \, d\tau + \int_{t}^{t+1} (t-\tau) \, d\tau
\]

\[
= \int_0^1 w \, dw + \int_0^1 w \, dw
\]

\[
= \frac{1}{2} + \frac{1}{2}
\]

\[
= 1.
\]

### The variance

The variance

\[
\text{Var}(\Delta y_t) = \mathbb{E}[\Delta y_t^2]
\]

\[
= \int_{t+1}^{t+2} (t+2-\tau)^2 \, d\tau + \int_{t}^{t+1} (t-\tau)^2 \, d\tau
\]

\[
= 2 \int_0^1 \tau^2 \, d\tau
\]

\[
= \frac{2}{3},
\]

The change in the average stock price has a smaller variance than the change in the stock price.

### First-Order Moving Average

The average stock price \(y_t\) is not a random walk. Its change is not white noise, but rather is a first-order moving average, with first-order autocorrelation \(\frac{1}{4}\).