Random Walk

According to the random-walk theory, a stock price follows a random walk. The change in the stock price is white noise.

In contrast, the average stock price during a period of time does not follow a random walk. Its change from one period to the next is a first-order moving average, with first-order autocorrelation $\frac{1}{4}$.

In the past, some econometric studies of the average stock price have wrongly claimed to have disproved the random walk theory,

Stock Price

Suppose that the stock price z_t is Wiener-Brownian motion.

The first-difference

$$\Delta z_t := z_{t+1} - z_t = \int_t^{t+1} \mathrm{d}z_\tau.$$

Its mean is zero.

Moving Average

Below we express variables as an integral (moving average) of the stock price changes $\mathrm{d}z_t$. We then calculate a variance or covariance using the relation

$$\mathrm{E}\left(\mathrm{d}z_t\,\mathrm{d}z_\tau\right) = \begin{cases} \mathrm{d}t \text{ if } t = \tau \\ 0 \text{ if } t \neq \tau. \end{cases}$$

The variance of the first difference

$$\operatorname{Var}(\Delta z_t) = \mathrm{E}\left[\left(\int_t^{t+1} \mathrm{d}z_\tau\right)^2\right] = \int_t^{t+1} \mathrm{E}\left[(\mathrm{d}z_\tau)^2\right] = \int_t^{t+1} \mathrm{d}\boldsymbol{\tau} = 1.$$

The first-order autocorrelation of Δz_t is zero, and the first-difference is white noise.

Average Stock Price

Let y_t denote the average stock price from time t to time $t+1$:

$$\begin{aligned} y_t &:= \int_t^{t+1} z_\tau \, \mathrm{d}\tau \\ &= \int_t^{t+1} \left(z_t + \int_t^{\tau} \mathrm{d}z_w \right) \mathrm{d}\tau \\ &= z_t + \int_t^{t+1} \left(\int_w^{t+1} \mathrm{d}\tau \right) \mathrm{d}z_w \\ &= z_t + \int_t^{t+1} (t+1-w) \, \mathrm{d}z_w . \end{aligned}$$

The average stock price is not a random walk.

Its first difference is

$$\begin{aligned}
\Delta y_t &:= y_{t+1} - y_t \\
&= \left[z_{t+1} + \int_{t+1}^{t+2} (t+2-w)\,\mathrm{d}z_w \right] \\
&\quad - \left[z_t + \int_{t}^{t+1} (t+1-w)\,\mathrm{d}z_w \right] \\
&= \int_{t}^{t+1} \mathrm{d}z_\tau + \int_{t+1}^{t+2} (t+2-\tau)\,\mathrm{d}z_\tau + \int_{t}^{t+1} (\tau - t - 1)\,\mathrm{d}z_\tau \\
&= \int_{t+1}^{t+2} (t+2-\tau)\,\mathrm{d}z_\tau + \int_{t}^{t+1} (\tau - t)\,\mathrm{d}z_\tau .
\end{aligned}$$

Weighted Average

The integration expresses the first difference as a weighted average of the stock price changes $\mathrm{d}z_\tau$, with all weights positive. The sum of the coefficients is one:

$$\begin{aligned}
\int_{t+1}^{t+2} (t+2-\tau)\,\mathrm{d}\tau + \int_{t}^{t+1} (\tau - t)\,\mathrm{d}\tau
&= \int_0^1 w\,\mathrm{d}w + \int_0^1 w\,\mathrm{d}w \\
&= \frac{1}{2} + \frac{1}{2} \\
&= 1.
\end{aligned}$$

The variance

$$\begin{aligned}\operatorname{Var}(\Delta y_t) &= \mathrm{E}\left[(\Delta y_t)^2\right] \\ &= \int_{t+1}^{t+2} (t+2-\tau)^2 \, \mathrm{d}\tau + \int_{t}^{t+1} (t-\tau)^2 \, \mathrm{d}\tau \\ &= 2\int_0^1 \tau^2 \, \mathrm{d}\tau \\ &= \frac{2}{3},\end{aligned}$$

The change in the average stock price has a smaller variance than the change in the stock price.

The first-order covariance

$$
\begin{aligned}
\operatorname{Cov}\left(\Delta y_{t+1}, \Delta y_t\right)
&= \mathrm{E}\left[\left(-\int_{t+1}^{t+2}(t+1-\tau)\,\mathrm{d}z_\tau\right)\left(\int_{t+1}^{t+2}(t+2-\tau)\,\mathrm{d}z_\tau\right)\right] \\
&= -\int_{t+1}^{t+2}(t+1-\tau)(t+2-\tau)\,\mathrm{d}\tau \\
&= \int_0^1 w(1-w)\,\mathrm{d}w \text{ (substituting } w := \tau-(t+1)) \\
&= \left(\frac{1}{2}w^2-\frac{1}{3}w^3\right)\Big]_0^1 \\
&= \frac{1}{6}.
\end{aligned}
$$

First-Order Moving Average

The average stock price y_t is not a random walk. Its change is not white noise, but rather is a first-order moving average, with first-order autocorrelation $\frac{1}{4}$.