Random Walk

According to the random-walk theory, a stock price follows a random walk. The change in the stock price is white noise.

In contrast, the average stock price during a period of time does not follow a random walk. Its change from one period to the next is a first-order moving average, with first-order autocorrelation $\frac{1}{4}$.

In the past, some econometric studies of the average stock price have wrongly claimed to have disproved the random walk theory,
Stock Price

Suppose that the stock price $z_t$ is Wiener-Brownian motion. The first-difference

$$\Delta z_t := z_{t+1} - z_t = \int_t^{t+1} \, dz_\tau.$$

Its mean is zero.
Moving Average

Below we express variables as an integral (moving average) of the stock price changes $dz_t$. We then calculate a variance or covariance using the relation

$$E (dz_t \ dz_\tau) = \begin{cases} 
    dt & \text{if } t = \tau \\
    0 & \text{if } t \neq \tau.
\end{cases}$$
The variance of the first difference

$$\text{Var} (\Delta z_t) = E \left[ \left( \int_t^{t+1} dz_\tau \right)^2 \right] = \int_t^{t+1} E \left[ (dz_\tau)^2 \right] = \int_t^{t+1} d\tau = 1.$$ 

The first-order autocorrelation of $\Delta z_t$ is zero, and the first-difference is white noise.
Average Stock Price

Let $y_t$ denote the average stock price from time $t$ to time $t+1$:

$$y_t := \int_t^{t+1} z_\tau \, d\tau$$

$$= \int_t^{t+1} \left( z_t + \int_t^\tau \, dz_w \right) \, d\tau$$

$$= z_t + \int_t^{t+1} \left( \int_w^{t+1} \, d\tau \right) \, dz_w$$

$$= z_t + \int_t^{t+1} \left( t + 1 - w \right) \, dz_w.$$

The average stock price is not a random walk.
Its first difference is

\[ \Delta y_t := y_{t+1} - y_t \]

\[ = \left[ z_{t+1} + \int_{t+1}^{t+2} (t + 2 - w) \, dz_w \right] - \left[ z_t + \int_{t}^{t+1} (t + 1 - w) \, dz_w \right] \]

\[ = \int_{t}^{t+1} d\tau \, + \int_{t+1}^{t+2} (t + 2 - \tau) \, d\tau \, + \int_{t}^{t+1} (\tau - t - 1) \, d\tau \]

\[ = \int_{t+1}^{t+2} (t + 2 - \tau) \, d\tau \, + \int_{t}^{t+1} (\tau - t) \, d\tau . \]
**Weighted Average**

The integration expresses the first difference as a weighted average of the stock price changes $d_z \tau$, with all weights positive. The sum of the coefficients is one:

$$
\int_{t+1}^{t+2} (t + 2 - \tau) \, d\tau + \int_{t}^{t+1} (\tau - t) \, d\tau
$$

$$
= \int_{0}^{1} w \, dw + \int_{0}^{1} w \, dw
$$

$$
= \frac{1}{2} + \frac{1}{2}
$$

$$
= 1.
$$
The variance

\[
\text{Var} (\Delta y_t) = \mathbb{E} \left[ (\Delta y_t)^2 \right]
\]

\[
= \int_{t+1}^{t+2} (t + 2 - \tau)^2 \, d\tau + \int_{t}^{t+1} (t - \tau)^2 \, d\tau
\]

\[
= 2 \int_{0}^{1} \tau^2 \, d\tau
\]

\[
= \frac{2}{3},
\]

The change in the average stock price has a smaller variance than the change in the stock price.
Financial Economics

The first-order covariance

\[ \text{Cov}(\Delta y_{t+1}, \Delta y_t) \]

\[ = E \left[ \left( - \int_{t+1}^{t+2} (t + 1 - \tau) \, d\tau \right) \left( \int_{t+1}^{t+2} (t + 2 - \tau) \, d\tau \right) \right] \]

\[ = - \int_{t+1}^{t+2} (t + 1 - \tau) (t + 2 - \tau) \, d\tau \]

\[ = \int_0^1 w (1 - w) \, dw \quad \text{(substituting } w := \tau - (t + 1)\text{)} \]

\[ = \left( \frac{1}{2} w^2 - \frac{1}{3} w^3 \right) \bigg|_0^1 \]

\[ = \frac{1}{6}. \]
First-Order Moving Average

The average stock price $y_t$ is not a random walk. Its change is not white noise, but rather is a first-order moving average, with first-order autocorrelation $\frac{1}{4}$. 