Two-State Model of Option Pricing

Rendleman and Bartter [1] put forward a simple two-state model of option pricing. As in the Black-Scholes model, to buy the stock and to sell the call in the hedge ratio obtains a risk-free portfolio. To avoid an opportunity for arbitrage profit, this portfolio must yield the risk-free rate of return, which sets the call price.

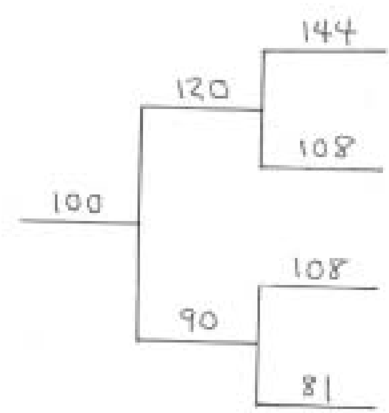
We analyze an example.

Stock

Consider a stock such that each period either the price rises by 20% or the price falls by 10%. Let S denote the stock price.

Suppose that the initial price in period zero is $S = 100$. The tree diagram in figure (1) shows the possibilities over two periods.

Figure 1: Stock Price



Risk-Free Asset

We assume that there exists a risk-free asset exists, with rate-of-return R.

For simplicity in the calculations below, assume $R = 0$: a one-dollar investment is worth just one dollar in the next period.

Call

Consider a call expiring in period two with exercise price 100. Let C denote the call price.

The stock price in period two determines the value of the call in period two. If the stock price in period two is 144, then the call price is $C = 144 - 100 = 44$. If the stock price is 108, then the value of the call is $C = 108 - 100 = 8$. If the stock price is 81, then it is not profitable to exercise the call, so its value is zero.

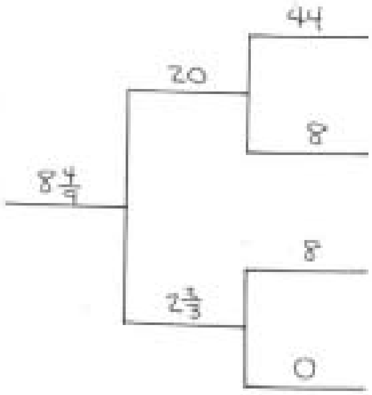
No Opportunity for Profitable Arbitrage

By working backwards in time, we can find the value of the call in the earlier periods. First one finds the two possible call prices in period one. Given these two values, one then finds the call price in period zero.

The key to the argument is that from the stock and the call one can create a risk-free portfolio. This portfolio must earn the risk-free rate of return, as otherwise there would be an opportunity for profitable arbitrage. From this condition, one can solve uniquely for the call price.

Figure (2) summarizes the results of the calculations explained below.

Figure 2: Call Price



Hedge Ratio

Suppose that the stock price in period one is 108, and compare the two possibilities in period two.

The stock price is either 144 or 108, so the difference is $\Delta S = 144 - 108 = 36$.

The call price is either 44 or 8, so the difference is $\Delta C = 44 - 8 = 36$.

The hedge ratio is therefore

$$\frac{\Delta C}{\Delta S} = \frac{36}{36} = 1.$$

Risk-Free Portfolio

Since the hedge ratio is one, to buy one share of the stock and to sell one call obtains a risk-free portfolio. If C is the price of the call in period one, then the cost of the portfolio is $120 - C$.

Let us verify that this portfolio is risk-free. If the stock price in period two is 144, the call price is 44, so the value of the portfolio is $144 - 44 = 100$. If the stock price in period two is 108, the call price is 8, so the value of the portfolio is $108 - 8 = 100$.

No Arbitrage

No opportunity for profitable arbitrage requires that this portfolio yield the risk-free rate of return:

$$(120 - C)(1 + R) = 100$$

is the no-arbitrage condition. Since $R = 0$, therefore $C = 20$.

Any other call price would allow profitable arbitrage.

Second Possibility in Period One

For a stock price of 90 in period one, we work out in the same way the call price in period one. Compare the two possibilities in period two. If the stock price is 108, then the call value is $108 - 100 = 8$. If the stock price is 81, then it is not profitable to exercise the call, so the call value is zero.

Hedge Ratio

As the stock price is either 108 or 81, the difference is $\Delta S = 108 - 81 = 27$. As the call price is either 8 or 0, the difference is $\Delta C = 8 - 0 = 8$.

The hedge ratio is therefore

$$\frac{\Delta C}{\Delta S} = \frac{8}{27}.$$

Risk-Free Portfolio

Hence to buy stock and to sell calls in the ratio of 8 to 27 obtains a risk-free portfolio. Suppose that one buys one share of stock and sells $27/8$ calls. If C is the price of the call in period one, then the cost of the portfolio is $90-(27/8)C$.

This portfolio is indeed risk-free. If the stock price in period two is 108, the value of the portfolio is $108-(27/8)\,8=81$. If the stock price in period two is 81, the value of the portfolio is $81-(27/8)\,0=81$.

No Arbitrage

No opportunity for profitable arbitrage requires that this portfolio yield the risk-free rate of return. Hence

$$[90-(27/8)C]\,(1+R)=81$$

is the no-arbitrage condition. Since $R=0$, therefore $C=2\frac{2}{3}$. Any other call price would allow profitable arbitrage.

Call Price in Period Zero

Working backwards, we work out in the same way the call price in period zero, when the stock price is 100.

In period one the stock price is either 120 or 90, so the difference is $\Delta S=120-90=30$. As the call price is either 20 or $2\frac{2}{3}$, the difference is $\Delta C=20-2\frac{2}{3}=17\frac{1}{3}$.

The hedge ratio is therefore

$$\frac{\Delta C}{\Delta S}=\frac{17\frac{1}{3}}{30}=\frac{26}{45}.$$

Risk-Free Portfolio

Hence to buy stock and to sell calls in the ratio of 26 to 45 obtains a risk-free portfolio. Suppose that one buys one share of stock and sells $45/26$ calls. If C is the price of the call in period zero, then the cost of the portfolio is $100-(45/26)C$.

This portfolio is indeed risk-free. If the stock price in period two is 120, the value of the portfolio is $120-(45/26)\,20=1110/13\approx 85.4$. If the stock price in period two is 90, the value of the portfolio is $90-(45/26)\left(2\frac{2}{3}\right)=1110/13\approx 85.4$.

No Arbitrage

No opportunity for profitable arbitrage requires that this portfolio yield the risk-free rate of return. Hence

$$[100-(45/26)C]\,(1+R)=1110/13$$

is the no-arbitrage condition. Since $R=0$, therefore $C=8\frac{4}{9}$. Any other call price would allow profitable arbitrage.

Changing Hedge Ratio

The hedge ratio changes as time passes.

Comparison with Black-Scholes

Note that the probability of each state is irrelevant to the calculations. This property corresponds to how in the Black-Scholes model the expected rate of return on the stock has no effect on the call price.

It is possible to obtain the Black-Scholes partial differential equation as the limit of the two-state model, by making the length of the time period shrink to zero.

References

[1] Richard J. Rendleman, Jr. and Brit J. Bartter. Two-state option pricing. *Journal of Finance*, XXXIV(5):1093–1110, December 1979. HG1J6.