Sensitivity of the Bond Price to the Interest Rate

How sensitive is the price of a bond to the market interest rate? If the market interest rate rises, does the price fall much, or only slightly?

The sensitivity is greater for a long-term bond than for a short-term bond.

One-Year Bond Versus Perpetual Bond

Consider a one-year bond with maturity value 1000 and coupon payment 100. The bond price is the present value

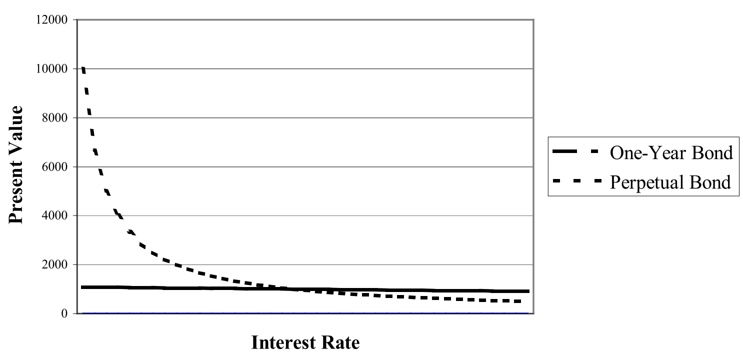
$$\frac{1000+100}{1+R}.$$

Also, consider a perpetual bond with coupon payment 100. The bond price is the present value

$$\frac{100}{R}.$$

The price of the perpetual bond is much more sensitive to the interest rate.

Figure 1: Present Value and Interest



Average Time-to-Payment

The owner of a bond receives coupon and principal payments, some sooner and some later. *Duration* is a measure of the average time-to-payment.

Duration determines the sensitivity of the price of a bond to the market interest rate:

$$\begin{aligned}&\%\text{ change in bond price} \approx \\ &\qquad -\text{duration} \times \%\text{ change in the interest rate}\end{aligned} \tag{1}$$

holds approximately.

For example, if the duration is five years, then a one per cent increase in the interest rate reduces the bond price by five per cent.

Consider a one-dollar payment n years in the future, with present value

$$\frac{1}{(1+R)^n}.$$

If the interest rate rises by 1%, then the present value falls by approximately n%. (The exact percentage change is slightly less.)

Here the duration is n years, so the relationship (1) holds approximately.

Duration

Definition 1 *Duration is the weighted average of the time to payment, using the present values as weights.*

Consider a four-year bond with the following payments. For the market interest rate 10%, the table shows the present value of each payment.

Time	Payment	Present Value	Weight	Time $\times$ Weight
1	100	91	0.091	0.09
2	100	83	0.083	0.17
3	100	75	0.075	0.23
4	1100	751	0.751	3.01
Total		1000	1.000	3.49

The duration is 3.49.

Interest Rate Increase

If the market interest rate rises from 10% to 11%, each payment falls in present value by the time to payment. For example, the payment at time 3 falls in present value by approximately 3%.

Hence the percentage decline in the total present value is the weighted average of time to payment, using the present values as weights.

If the interest rate rises 1%, then the duration is the approximate percentage decline in the present value.

		$R = .10$	$R = .11$	Percentage
Time	Payment	PV	PV	Decline
1	100	91	90	0.90
2	100	83	81	1.79
3	100	75	73	2.68
4	1100	751	725	3.56
Total		1000	969	3.10

Calculus Derivation

Suppose that the interest rate changes by ΔR and the present value changes by ΔPV. The fractional change in the present value divided by the change in the interest rate is

$$\frac{\frac{\Delta PV}{PV}}{\Delta R}.$$

For example, if this ratio is -4, then a one per cent increase in the interest rate will reduce the present value by 4 per cent.

Derivative

Letting the change in the interest rate shrink toward zero, in the limit the ratio is expressed by the derivative,

$$\lim_{\Delta R \to 0} \frac{\frac{\Delta PV}{PV}}{\Delta R} = \frac{1}{PV} \frac{\mathrm{d}PV}{\mathrm{d}R}.$$

Consider the one-dollar payment n years in the future, with present value

$$PV = \frac{1}{(1+R)^n}.$$

Taking the derivative,

$$\begin{aligned}\frac{1}{PV}\frac{\mathrm{d}PV}{\mathrm{d}R} &= \frac{1}{\left[\frac{1}{(1+R)^n}\right]}\frac{\mathrm{d}\left[\frac{1}{(1+R)^n}\right]}{\mathrm{d}R} \\ &= (1+R)^n\left[\frac{-n}{(1+R)^{n+1}}\right] \\ &= -\frac{n}{(1+R)}.\end{aligned}$$

Improved Approximation

This equation says that a one per cent increase in the interest rate changes the present value by approximately

$$-\frac{n}{(1+R)}$$

per cent. The division by $1+R$ gives a more accurate value for the percentage decline in the present value.

Duration and Present Value

For an arbitrary bond with payments at different times, a similar calculation yields

$$\frac{1}{PV}\frac{\mathrm{d}PV}{\mathrm{d}R} = -\frac{D}{(1+R)},$$

in which D is the duration.

For the four-year bond example, dividing $D = 3.49$ by $1+R = 1.10$ gives $3.49/1.10 = 3.17$. This figure is close to the actual 3.10 per cent decline in the present value.