Perpetual Bond

Consider a perpetual bond promising to pay one dollar interest per year.

Discounted at the interest rate R, the present value of the interest payments is

$$PV = \frac{1}{R}.$$

Yield to Maturity

The yield to maturity is the rate of return from holding the bond to maturity, if there is no default and all payments are made as promised.

Consequently the yield to maturity is the discount rate that makes the present value of promised payments equal to the price.

Thus the yield to maturity on the perpetual bond is

$$\frac{1}{P}.$$

Default Risk

We analyze the perpetual bond when there is some chance of default.

Let R denote the market interest rate.

A chance of default makes the bond worth less, so then

$$P < \frac{1}{R}.$$

The yield to maturity is then greater than R, as investors demand a higher yield to compensate for the chance of default.

Simple Model of Default

Let us analyze the simple model in which each year there is a constant probability Π of default.

The bond pays one dollar interest per year until it defaults; the bond price stays constant at P. When the bond defaults, no further payments are ever made, so the bond price falls to zero.

We solve for the equilibrium market price of the bond.

Asset-Market Equilibrium

Two equivalent conditions for asset-market equilibrium are:

Rate of Return The expected rate of return equals the market interest rate;

Present Value The asset price equals the present value of current and expected future payments.

For our simple model with default, we show that both conditions yield the same value for the equilibrium market price.

Rate-of-Return Condition

Using the rate-of-return condition, let us determine the market equilibrium price of the bond.

Each year, with probability $1 - \Pi$, the bond will not default, and its market price stays at P. The return on the bond is just the interest, so the rate of return is the fraction

$$\frac{1}{P}.$$

With probability Π, the bond defaults and becomes worthless. The rate of return is -100%; as a fraction,

$$-1.$$

Expected Rate of Return

For a random variable, the expected value is a weighted average of the values, using the probabilities as weights. The expected value is a measure of the average value.

The equilibrium condition that the expected rate of return equals the market interest rate is

$$(1-\Pi)\left(\frac{1}{P}\right)+\Pi(-1)=R.$$

Solving for P gives

$$P=\frac{1-\Pi}{R+\Pi}. \tag{1}$$

If the probability Π of default is zero, then the formula (1) reduces to the standard formula

$$P = \frac{1}{R}$$

for the value of a perpetual bond.

If the probability is positive, the price is lower. As the probability of default rises, the price falls. Equivalently, the yield to maturity rises.

As the probability of default approaches one, then the price approaches zero.

Present-Value Condition

In market equilibrium, the market price must be the present value of expected payments. We show that this present-value condition obtains the same formula (1) for the price.

We calculate the present value of expected payments.

Probability of Default

After n years have passed, what is the probability that the bond has defaulted and will make no more payments?

Consider the case $\Pi = \frac{1}{2}$; each year there is a 50% chance of default.

After one year, there is a 50% chance that the bond will be in default, and a 50% chance that it will not be in default.

For the latter, there is then a 50% chance that the bond will go into default in the second year. Thus the probability that the bond will not be in default after two years is

$$\frac{1}{2} \times \frac{1}{2} = \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

(25% chance of no default). The probability that the bond will be in default is

$$1 - \frac{1}{4} = \frac{3}{4}.$$

After n years, the probability that the bond will still be paying interest is

$$\left(\frac{1}{2}\right)^n. \tag{2}$$

The expected payment in year n is simply this probability (2).

The present value of the expected payments is

$$\frac{\left(\frac{1}{2}\right)}{1+R}+\frac{\left(\frac{1}{2}\right)^2}{(1+R)^2}+\frac{\left(\frac{1}{2}\right)^3}{(1+R)^3}+\cdots.$$

General Formula

For an arbitrary probability Π, the probability that the bond will not be in default after n years is

$$(1-\Pi)^n.$$

This value is the expected payment in year n.

Present Value of Expected Payments

The present value of expected payments is therefore

$$PV = \frac{1-\Pi}{1+R} + \frac{(1-\Pi)^2}{(1+R)^2} + \frac{(1-\Pi)^3}{(1+R)^3} + \cdots . \tag{3}$$

A higher Π raises the probability of default and reduces the present value.

Infinite Geometric Sum

The present value is an infinite geometric sum, taking the form

$$\frac{a}{1-b} = a + ab + ab^2 + \cdots .$$

In the present value (3),

$$a = \frac{1-\Pi}{1+R}$$
$$b = \frac{1-\Pi}{1+R}.$$

Since $b < 1$, the terms in the infinite sum become smaller, and the infinite sum is finite.

$$\begin{aligned}
PV &= \frac{a}{1-b} \\
&= \frac{\left(\frac{1-\Pi}{1+R}\right)}{1-\left(\frac{1-\Pi}{1+R}\right)} \\
&= \frac{\frac{1-\Pi}{1+R}}{\frac{1+R}{1+R}-\frac{1-\Pi}{1+R}} \\
&= \frac{1-\Pi}{(1+R)-(1-\Pi)} \\
&= \frac{1-\Pi}{R+\Pi},
\end{aligned}$$

the same as (1).

Equivalent Equilibrium Conditions

The rate-of-return condition and the present-value condition are equivalent; either implies the other.

When the price equals the present value, we know that the expected rate of return is R.