

Call Price as a Function of the Stock Price

Intuitively, the call price should be an increasing function of the stock price.

This relationship allows one to develop a theory of option pricing, derived from the absence of profitable arbitrage.

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Hedge Ratio

For example, suppose that the call price rises one dollar when the stock price rises two dollars. One refers to this one-to-two ratio as the *hedge ratio*.

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Risk-Free Portfolio

Suppose that one buys one share of stock and sells two calls. Since the hedge ratio is $1/2$, this portfolio is *risk-free*. Any gain on the stock is offset by a loss on the calls, and *vice versa*.

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No Arbitrage

To rule out an opportunity for profitable arbitrage, the rate of return on this portfolio must be the risk-free return.

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Black-Scholes Model

Black and Scholes [1] derive a formula for the call price via this route. They assume that the stock price follows a random walk, with a constant mean and variance of the rate of return.

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Model Parameters

The variables and parameters affecting the call price are:

- Stock price
- Time to expiration
- Striking price
- Variance of the rate of return
- Mean rate of return
- Risk-free rate of return.

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Risk-Free Portfolio

The idea is to solve for the call price C as a function of the stock price S and the time to expiration T . The function $C(S, T)$ is then determined by the parameters.

The hedge ratio $\frac{\partial C}{\partial S}$ may change as time passes. Nevertheless, at any point in time, the hedge ratio determines a risk-free portfolio, in which one buys one share of stock and sells

$$\frac{1}{\frac{\partial C}{\partial S}}$$

calls.

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Partial Differential Equation

The condition that this portfolio must earn the risk-free return then determines a partial differential equation that the call price function must satisfy. The value of the call at the expiration date gives a boundary condition, and one solves the partial differential equation to obtain the Black-Scholes formula. One works backwards in time from the value at expiration to determine the call price at earlier times.

The technique uses stochastic calculus, a generalization of calculus adding random errors.

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Time to Expiration and Striking Price

From a previous no-arbitrage argument, we know that the call price rises as the time to expiration increases or as the striking price falls. The Black-Scholes formula does satisfy this property.

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Stock Price

As the stock price rises, the call price rises.

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Variance

A higher variance *raises* the call price.

This property is obvious for an out-of-the-money option: higher variance increases the likelihood of profitable exercise.

The property also holds for an in-the-money option. A higher variance does increase the chance of a big decline in the stock price, so the probability that the option will not be exercised increases. However a higher variance also increases the potential for a big profit.

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Mean Rate of Return

Perhaps surprisingly, the mean rate of return on the stock has *no effect* on the call price.

A higher mean does imply a greater chance of profitable arbitrage, so the expected profit from arbitrage rises.

However a high mean also implies that these profits should be discounted at a higher rate, and it turns out that this discount effect exactly offsets the higher expected profit.

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Risk-Free Rate of Return

An increase in the risk-free return lowers the call price.

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Appendix The Black-Scholes Option Pricing Formula

Notation

S	Stock price
C	Call price
X	Exercise price
R	Risk-free rate of return
M	Stock return risk premium
V	Stock return variance
T	Time to expiration

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Black-Scholes Formula

Solution 1 (Black-Scholes Option Pricing Formula)

$$C(S, T) = SN \left[\frac{\ln(S/X) + (R + V/2)T}{\sqrt{TV}} \right] - Xe^{-RT} N \left[\frac{\ln(S/X) + (R - V/2)T}{\sqrt{TV}} \right].$$

Here $N(v)$ is the cumulative unit normal, the probability that the value is less than or equal to v .

Note that M does not appear in the formula.

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References

- [1] Fischer Black and Myron Scholes. The pricing of options and corporate liabilities. *Journal of Political Economy*, 81(3):637–654, May/June 1973. HB1J7. 5

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