Call Price as a Function of the Stock Price

Intuitively, the call price should be an increasing function of the stock price.

This relationship allows one to develop a theory of option pricing, derived from the absence of profitable arbitrage.

Hedge Ratio

For example, suppose that the call price rises one dollar when the stock price rises two dollars. One refers to this one-to-two ratio as the *hedge ratio*.

Risk-Free Portfolio

Suppose that one buys one share of stock and sells two calls. Since the hedge ratio is $1/2$, this portfolio is *risk-free*. Any gain on the stock is offset by a loss on the calls, and *vice versa*.

No Arbitrage

To rule out an opportunity for profitable arbitrage, the rate of return on this portfolio must be the risk-free return.

Black-Scholes Model

Black and Scholes [1] derive a formula for the call price via this route. They assume that the stock price follows a random walk, with a constant mean and variance of the rate of return.

Model Parameters

The variables and parameters affecting the call price are:

- Stock price
- Time to expiration
- Striking price
- Variance of the rate of return
- Mean rate of return
- Risk-free rate of return.

Risk-Free Portfolio

The idea is to solve for the call price C as a function of the stock price S and the time to expiration T. The function $C(S,T)$ is then determined by the parameters.

The hedge ratio $\frac{\partial C}{\partial S}$ may change as time passes. Nevertheless, at any point in time, the hedge ratio determines a risk-free portfolio, in which one buys one share of stock and sells

$$\frac{1}{\frac{\partial C}{\partial S}}$$

calls.

Partial Differential Equation

The condition that this portfolio must earn the risk-free return then determines a partial differential equation that the call price function must satisfy. The value of the call at the expiration date gives a boundary condition, and one solves the partial differential equation to obtain the Black-Scholes formula. One works backwards in time from the value at expiration to determine the call price at earlier times.

The technique uses stochastic calculus, a generalization of calculus adding random errors.

Time to Expiration and Striking Price

From a previous no-arbitrage argument, we know that the call price rises as the time to expiration increases or as the striking price falls. The Black-Scholes formula does satisfy this property.

Stock Price

As the stock price rises, the call price rises.

Variance

A higher variance *raises* the call price.

This property is obvious for an out-of-the-money option: higher variance increases the likelihood of profitable exercise.

The property also holds for an in-the-money option. A higher variance does increase the chance of a big decline in the stock price, so the probability that the option will not be exercised increases. However a higher variance also increases the potential for a big profit.

Mean Rate of Return

Perhaps surprisingly, the mean rate of return on the stock has *no effect* on the call price.

A higher mean does imply a greater chance of profitable arbitrage, so the expected profit from arbitrage rises.

However a high mean also implies that these profits should be discounted at a higher rate, and it turns out that this discount effect exactly offsets the higher expected profit.

Risk-Free Rate of Return

An increase in the risk-free return lowers the call price.

Appendix
The Black-Scholes Option Pricing Formula

Notation

S	Stock price
C	Call price
X	Exercise price
R	Risk-free rate of return
M	Stock return risk premium
V	Stock return variance
T	Time to expiration

Black-Scholes Formula

Solution 1 (Black-Scholes Option Pricing Formula)

$$C(S,T) = S\,\mathrm{N}\left[\frac{\ln(S/X) + (R + V/2)\,T}{\sqrt{TV}}\right] - X\mathrm{e}^{-RT}\,\mathrm{N}\left[\frac{\ln(S/X) + (R - V/2)\,T}{\sqrt{TV}}\right].$$

Here $\mathrm{N}(v)$ is the cumulative unit normal, the probability that the value is less than or equal to v.

Note that M does not appear in the formula.

References

[1] Fischer Black and Myron Scholes. The pricing of options and corporate liabilities. *Journal of Political Economy*, 81(3):637–654, May/June 1973. HB1J7. 5