Asset-Market Equilibrium

Asset-market equilibrium means that demand equals supply for an asset.

Two Equivalent Conditions

Economic theory puts forward two equivalent conditions for asset-market equilibrium:

Rate of Return The rate of return equals the market interest rate;

Present Value The asset price equals the present value of current and future payments.

We explain these two conditions and show that they are equivalent—either condition implies the other.

(For simplicity, we assume that there is no uncertainty.)

Rate of Return

The *rate of return* is the profit divided by the amount invested.

Consider an asset with payment stream $\$_t$ at time t. For a stock, the payment would be the dividend. For a bond, the payment would be interest or principal. Let P_t denote the price at time t. For simplicity, take P_t as the price just after the payment $\$_t$ has been made.

Consider an investment at time $t-1$. The profit is the payment $\$_t$ plus the capital gain $P_t - P_{t-1}$. The rate of return during period t is the profit divided by the beginning-of-period price P_{t-1},

$$\frac{\$_t + (P_t - P_{t-1})}{P_{t-1}}.$$

Rate-of-Return Equilibrium Condition

The rate-of-return condition for asset-market equilibrium states that the rate of return equals the market interest rate:

$$\frac{\$_t + (P_t - P_{t-1})}{P_{t-1}} = R. \tag{1}$$

The *market interest rate* refers to the rate of return R that can be obtained elsewhere.

Disequilibrium

If the rate of return were greater than the market interest rate, the security would be seen as a “good buy.” Investors would like to buy the security; those holding the security would not want to sell it. Demand would exceed supply. The reverse inequality would lead to excess supply.

Present Value

The *present value* of the current and expected future payments on an asset refers to the dollars attainable now by borrowing against the future payments.

Example

Consider an asset paying 110 at time 1 ($\$_1 = 110$, $\$_t = 0$ for $t \neq 1$), and suppose that the interest rate is .10. The present value of the payment at time 0 is

$$PV = \frac{\$_1}{1+R} = \frac{110}{1.10} = 100.$$

At time 0, one can borrow 100 against the payment stream; at time 1, one uses the payment 110 to pay off the loan plus interest.

Analogously, consider an asset paying 121 at time 2 ($\$_2 = 121$, $\$_t = 0$ for $t \neq 2$), and suppose that the interest rate is .10. The present value of the payment at time 0 is

$$PV = \frac{\$_2}{(1+R)^2} = \frac{121}{(1.10)^2} = 100.$$

At time 0, one can borrow 100 against the payment stream. At time 1, one owes 110, including interest 10; at time 2, one owes 121, which equals the principal 110 plus interest 11.

Formula for Present Value

In general, the present value is given by the following formula:

$$PV = \frac{\$_1}{1+R} + \frac{\$_2}{(1+R)^2} + \frac{\$_3}{(1+R)^3} + \cdots . \tag{2}$$

Present-Value Equilibrium Condition

The present-value condition for asset-market equilibrium is that the asset price equals the present value (2) of the payments:

$$P_t = \frac{\$_{t+1}}{1+R} + \frac{\$_{t+2}}{(1+R)^2} + \frac{\$_{t+3}}{(1+R)^3} + \cdots . \tag{3}$$

Intuitively, the present value is the worth of the asset. If the price is less than the present value, then there is excess demand for the asset. The reverse inequality means excess supply.

Equivalence

The present-value condition (3) is equivalent to the equilibrium condition that the rate of return must equal the market interest rate. We first demonstrate the equivalence in several examples.

Consider first the simple asset defined above, paying 110 at time 1 ($\$_1 = 110$, $\$_t = 0$ for $t \neq 1$), with the interest rate at .10. A fair price for the asset at time 0 is 100; one then obtains rate of return .10. But 100 is the present value of the payments.

Perpetual Bond

Consider next a *perpetual bond*, a bond paying one-dollar interest per period in perpetuity; the principal is never repaid. If the interest rate is R, what is a fair price for the bond?

A fair price is

$$\frac{1}{R},$$

for the bond then has rate of return R. For example, if $R = .10$, then the bond should sell for \$10, so it will return 10%. A lower price would give a higher yield, and a higher price would give a lower yield.

To show the equivalence between the two equilibrium conditions, we must show that the present value of the payments is $1/R$. The present value is

$$PV = \frac{1}{1+R} + \frac{1}{(1+R)^2} + \frac{1}{(1+R)^3} + \cdots, \tag{4}$$

(the first payment comes at time 1). We need to evaluate this sum.

Infinite Geometric Sum

We derive a general formula for this type of expression, used often in economic theory. Consider the *infinite geometric sum*

$$x = a + ab + ab^2 + ab^3 + \cdots .$$

Each term is just b times the previous term. Here a is the first term, and b is the ratio of successive terms.

To find x, multiply both sides of the equation by b:

$$bx = ab + ab^2 + ab^3 + \cdots .$$

Subtracting the second equation from the first gives

$$\begin{aligned}(1-b)x &= \left(a + ab + ab^2 + ab^3 + \cdots\right) \\ &\quad - \left(ab + ab^2 + ab^3 + \cdots\right) \\ &= a.\end{aligned}$$

The other terms cancel, for we have an infinite number of terms that cancel.

Hence

$$x = \frac{a}{1-b}.$$

The formula is valid as long as $|b| < 1$. If b were greater than one, then the successive terms in the sum would become larger; and the sum would be infinite.

Present Value for the Perpetual Bond

We apply this formula to the present value for the perpetual bond. The first term is $a = 1/(1+R)$ and the ratio of successive terms is $b = 1/(1+R)$. Hence

$$PV = \frac{\frac{1}{1+R}}{\left(1 - \frac{1}{1+R}\right)} = \frac{\frac{1}{1+R}}{\left(\frac{1+R}{1+R} - \frac{1}{1+R}\right)} = \frac{1}{R}.$$

Stock

Next we analyze a related but more complex example.

Consider a stock paying current dividend D, and the dividend grows at rate G ($\$_1 = D$, $\$_2 = D(1+G)$, etc.).

What is a fair price P for the stock at time 0?

The rate of return is the dividend yield plus the rate of capital gain. The dividend yield is D/P. Since the dividend grows each period at rate G, intuitively the stock price should also grow at this rate, so the rate of capital gain is G. The rate of return is therefore

$$\frac{D}{P}+G.$$

Setting the rate of return equal to the market interest rate gives

$$\frac{D}{P} + G = R.$$

Solving for P yields

$$P = \frac{D}{R - G}. \tag{5}$$

The formula makes sense qualitatively: raising D, reducing R, or raising G should increase P.

Present Value for the Stock

We show that this price equals the present value of the payments:

$$PV = \frac{D}{1+R} + \frac{D(1+G)}{(1+R)^2} + \frac{D(1+G)^2}{(1+R)^3} + \cdots .$$

This infinite sum is geometric, with first term $a = D/(1+R)$ and ratio of successive terms $b = (1+G)/(1+R)$. Hence

$$PV = \frac{\frac{D}{1+R}}{\left(1 - \frac{1+G}{1+R}\right)} = \frac{\frac{D}{1+R}}{\left(\frac{1+R}{1+R} - \frac{1+G}{1+R}\right)} = \frac{D}{R-G},$$

in agreement with (5).

The sum is finite as long as $R > G$. If $G > R$, then the successive terms in the sum would become larger; and the sum would be infinite.

General Equivalence

In general, the two conditions for equilibrium are equivalent. If the price equals the present value at every moment, then the rate of return equals the market interest rate at every moment; and *vice versa*. We prove the equivalence.

Assume that the price equals the present value at time 0:

$$P_0 = \frac{\$_1}{1+R} + \frac{\$_2}{(1+R)^2} + \frac{\$_3}{(1+R)^3} + \cdots. \tag{6}$$

At time 1, the price also equals the present value:

$$P_1 = \frac{\$_2}{(1+R)} + \frac{\$_3}{(1+R)^2} + \cdots. \tag{7}$$

Here the present value is discounted to time 1, not to time 0. Consider, for example, the first term. The payment $\$_2$ at time 2 is only one period into the future, so its present value at time 1 is the payment discounted by $1+R$.

From (6) and (7), we show that the rate of return equals the market interest rate. Dividing (7) by $1+R$ gives

$$\frac{P_1}{1+R} = \frac{\$_2}{(1+R)^2} + \frac{\$_3}{(1+R)^3} + \cdots .$$

Subtracting this expression from (6) gives

$$P_0 - \frac{P_1}{1+R} = \frac{\$_1}{1+R}.$$

Multiplying by $1+R$ and rearranging gives

$$\frac{\$_1+(P_1-P_0)}{P_0}=R;$$

the rate of return equals the market interest rate. And working backwards shows that this condition implies that price equals present value.

Uncertainty

With uncertainty, the two equilibrium conditions involve expected values (average values):

Rate of Return The expected rate of return equals the market interest rate;

Present Value The asset price equals the present value of current and expected future payments.