Fractional Change

Consider a variable $x$ that changes by $\Delta x$ during a time period of length $\Delta t$.

The *fractional change* is

$$\frac{\Delta x}{x}$$

For example, a fractional change of .05 would mean an increase of 5%.
Growth Rate

The *growth rate* is the fractional change per unit time,

\[ \frac{\Delta x}{x} \div \frac{\Delta t}{\Delta t}, \]

the fractional change divided by the length of the time period. For example, suppose that the length of the time period is one quarter year, \( \Delta t = .25 \). And suppose that the fractional change in the variable is .02. Then the growth rate is

\[ \frac{.02}{.25} = .08, \]

a rate of 8% per year.
Growth Rate in Continuous Time

The growth rate in continuous time is the growth rate as the length of the period shrinks to zero, $\Delta t \to 0$. In the limit, the growth rate is the derivative,

$$\lim_{\Delta t \to 0} \frac{1}{x} \frac{\Delta x}{\Delta t} = \frac{1}{x} \frac{dx}{dt}.$$
Derivative of the Logarithm

The growth rate is also the derivative of the logarithm,

\[
\frac{d \ln x}{dt} = \frac{1}{x} \frac{dx}{dt}.
\]

For a small changes, the change in the logarithm must be the fractional change,

\[
\Delta \ln x \approx \frac{\Delta x}{x}.
\]
Constant Growth Rate

If the variable grows at the constant rate $g$,

$$\ln x_t = \ln x_0 + gt.$$ 

Taking the exponent shows

$$x_t = e^{\ln x_t} = e^{(\ln x_0 + gt)} = e^{\ln x_0} e^{gt} = x_0 e^{gt}.$$ 

Thus

$$e^{gt}$$

represents constant growth at rate $g$. 

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Graph

A non-logarithmic graph of $x_t = x_0e^{gt}$ against time has a steadily increasing slope.

A logarithmic graph of $\ln x_t = \ln (x_0e^{gt}) = \ln x_0 + gt$ against time has the constant slope $g$. Equal upward and downward shifts along the curve represent equal upward and downward percentage changes in the variable.
Growth Rate of a Product

The growth rate of a product is the sum of the growth rates.
For example, nominal income $Y$ is the price level $P$ times real income $y$,

$$Y = Py.$$
Then the growth rate of nominal income is the inflation rate plus the growth rate of real income,

\[
\frac{1}{Y} \frac{dY}{dt} = \frac{d \ln Y}{dt} = \frac{d \ln Py}{dt} = \frac{d (\ln P + \ln y)}{dt} = \frac{d \ln P}{dt} + \frac{d \ln y}{dt} = \frac{1}{P} \frac{dP}{dt} + \frac{1}{y} \frac{dy}{dt}.
\]
Growth Rate of a Ratio

Analogously, the growth rate of a ratio is the growth rate of the numerator minus the growth rate of the denominator.

For example, the growth rate of the real wage $\frac{W}{P}$ is the growth rate of the nominal wage $W$ minus the growth rate of the nominal price level $P$. 