Neoclassical One-Sector Growth Model

Consider the Solow neoclassical one-sector growth model with Cobb-Douglas production function

$$Y = F(K, L) = K^{\frac{1}{3}} L^{\frac{2}{3}}.$$

Gross saving is sY, with $s = .12$. The rate of population growth $n = .03$. Initially the capital/labor ratio $k = K/L = 4$.

Intensive Production Function

Because returns to scale are constant, output *per capita* is

$$\frac{F(K,L)}{L} = F\left(\frac{K}{L}, \frac{L}{L}\right) = F(k,1) = f(k).$$

Applying this relationship to the production function here,

$$f(k) = F(k,1) = k^{\frac{1}{3}} 1^{\frac{2}{3}} = k^{\frac{1}{3}}.$$

Change in the Capital/Labor Ratio

Expressed *per capita*, capital deepening $\mathrm{d}k/\mathrm{d}t$ equals saving $sf(k)$ less capital widening nk:

$$\frac{\mathrm{d}k}{\mathrm{d}t} = sf(k) - nk.$$

The capital widening $\mathrm{d}k/\mathrm{d}t$ is the increase in capital *per capita*.

Since national income equals national product, income *per capita* equals output *per capita* $f(k)$. Saving *per capita* $sf(k)$ is income *per capita* times the fraction of income saved.

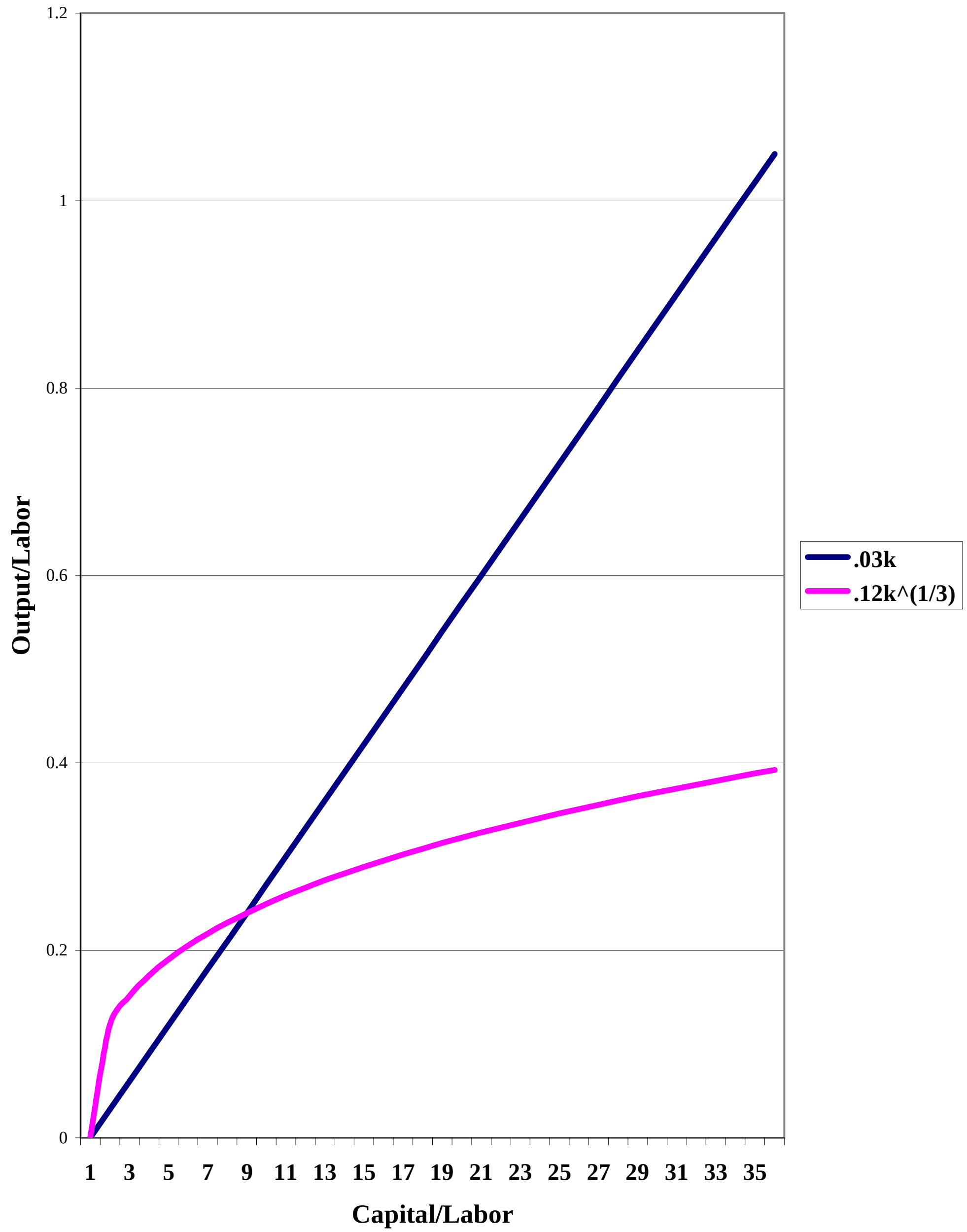
Part of the saving is used to equip new workers with capital. The population growth rate n is the number of new workers *per capita*. Each worker requires k units of capital, so saving *per capita* for capital widening is nk.

The remainder of the saving is available for capital deepening, increasing the capital *per capita*. This residual saving *per capita* is $sf(k) - nk$, so capital *per capita* goes up by this amount.

For the model here,

$$\frac{\mathrm{d}k}{\mathrm{d}t} = sf(k) - nk = .12k^{\frac{1}{3}} - .03k,$$

and the Solow diagram shows the relationship.



1.2
1
0.8
0.6
0.4
0.2
0
Output/Labor
1 3 5 7 9 11 13 15 17 19 21 23 25 27 29 31 33 35
Capital/Labor
.03k
.12k^(1/3)

Short-Run Behavior

For the initial value $k = 4$, the figure shows that saving exceeds capital widening, so capital deepening occurs. The capital/labor ratio rises:

$$\begin{aligned} \frac{\mathrm{d}k}{\mathrm{d}t} &= .12k^{\frac{1}{3}} - .03k \\ &= .12\,(4)^{\frac{1}{3}} - .03 \times 4 \\ &\approx .07. \end{aligned}$$

Long-Run Behavior

In the long run, the economy converges to steady-state growth. The capital/labor ratio is constant:

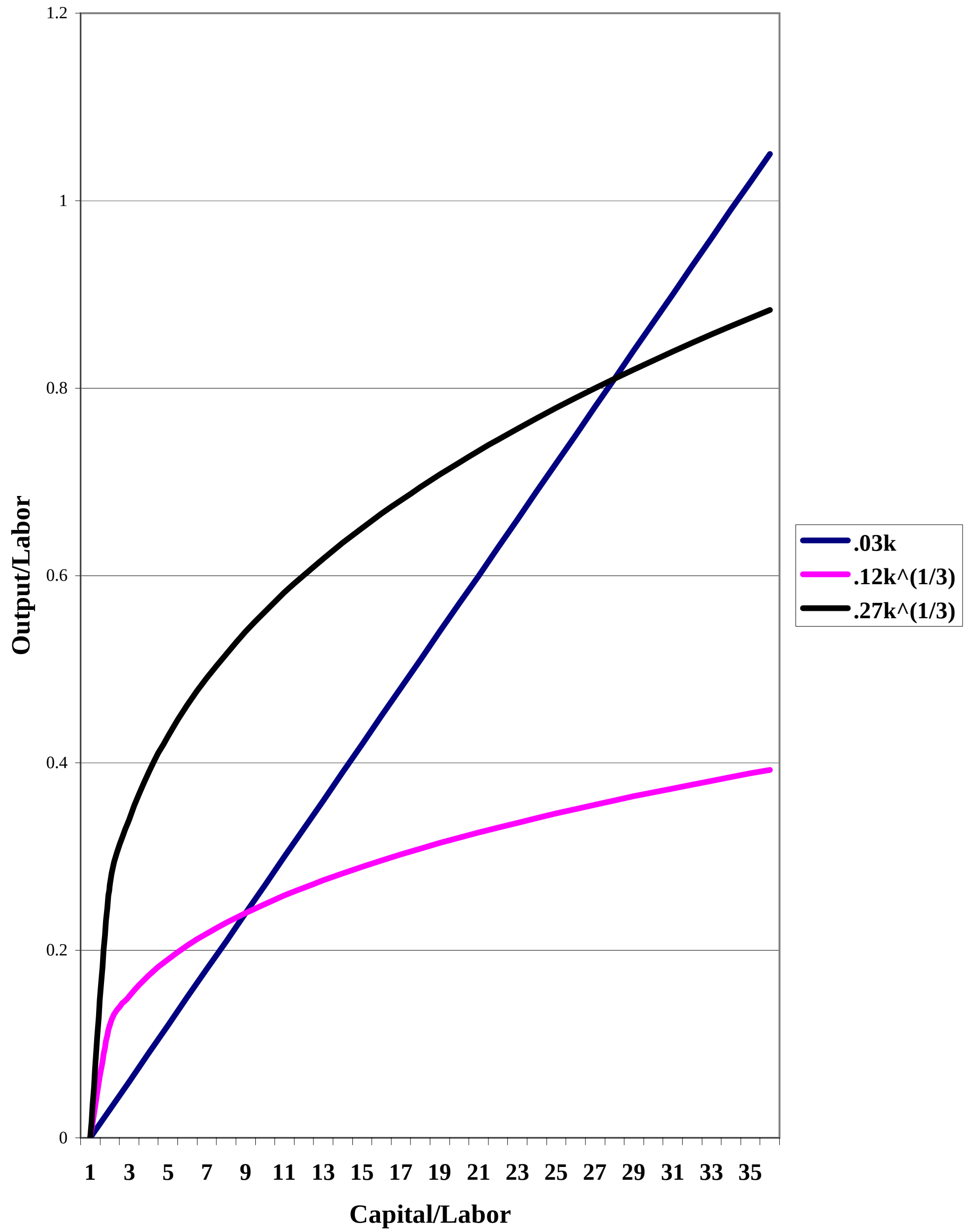
$$0 = \frac{\mathrm{d}k}{\mathrm{d}t} = sf(k) - nk = .12k^{\frac{1}{3}} - .03k,$$

with solution $k = 8$.

Per capita values are constant. The growth rate of output, capital, consumption, and investment are all constant at the rate of population growth, $n = .03$.

Higher Saving Rate

Alternatively, suppose that the saving fraction is $s = .27$. In the Solow diagram, saving *per capita* rises.



1.2
1
0.8
0.6
0.4
0.2
0
Output/Labor
1 3 5 7 9 11 13 15 17 19 21 23 25 27 29 31 33 35
Capital/Labor
.03k
.12k^(1/3)
.27k^(1/3)

Short-Run Behavior

In the short run, saving is higher. The capital/labor ratio increases more rapidly, and higher saving and investment cause faster output growth.

For the initial value $k = 4$,

$$\begin{aligned}\frac{\mathrm{d}k}{\mathrm{d}t} &= .27k^{\frac{1}{3}} - .03k \\ &= .27\left(4\right)^{\frac{1}{3}} - .03 \times 4 \\ &\approx .31.\end{aligned}$$

Long-Run Behavior

In the long run, the economy again converges to steady-state growth, but the capital/labor ratio is higher. In steady-state growth,

$$0 = \frac{\mathrm{d}k}{\mathrm{d}t} = sf(k) - nk = .27k^{\frac{1}{3}} - .03k,$$

with solution $k = 27$.

Per capita values are constant, but output *per capita* is higher with higher saving. Again the population growth $n = .03$ determines the growth rate of output, capital, consumption, and investment.