Infinite Geometric Sum

We derive a general formula for the value of an *infinite geometric sum*, an expression of the form

$$x = a + ab + ab^2 + ab^3 + \cdots . \tag{1}$$

Each term is just b times the previous term. Here a is the first term, and b is the ratio of successive terms.

To find x, multiply both sides of the equation by b:

$$bx = ab + ab^2 + ab^3 + \cdots . \tag{2}$$

Subtracting equation (2) from equation (1) gives

$$\begin{aligned}(1-b)x &= \left(a + ab + ab^2 + ab^3 + \cdots\right) \\ &\quad - \left(ab + ab^2 + ab^3 + \cdots\right) \\ &= a.\end{aligned}$$

The other terms cancel, for we have an infinite number of terms that cancel.

Hence

$$x = \frac{a}{1-b}. \tag{3}$$

The formula is valid as long as $|b| < 1$. If b were greater than one, then the successive terms in the sum would become larger; and the sum would be infinite.

Numerical Example

Consider a numerical example. Evaluate

$$1+\frac{1}{2}+(\frac{1}{2})^2+(\frac{1}{2})^3+\cdots.$$

Term-by-term, the total rises as $1.000, 1.500, 1.750, 1.875, \ldots$.

Here

$$a = 1$$

$$b = \frac{1}{2}.$$

Substituting into the formula (3), the value is

$$\frac{a}{1-b} = \frac{1}{1-\frac{1}{2}} = \frac{1}{\frac{1}{2}} = 2.$$

Applications

The formula for an infinite geometric sum has repeated application in economics.

- Macroeconomics: the multiplier effect;
- Monetary Economics: the money multiplier;
- Financial Economics: present value.