Fractional Change

Consider a variable x that changes by Δx during a time period of length Δt.

The *fractional change* is

$$\frac{\Delta x}{x}$$

For example, a fractional change of .05 would mean an increase of 5%.

Growth Rate

The *growth rate* is the fractional change per unit time,

$$\frac{\frac{\Delta x}{x}}{\Delta t},$$

the fractional change divided by the length of the time period.

For example, suppose that the length of the time period is one quarter year, $\Delta t = .25$. And suppose that the fractional change in the variable is .02. Then the growth rate is

$$\frac{.02}{.25} = .08,$$

a rate of 8% per year.

Growth Rate in Continuous Time

The growth rate in continuous time is the growth rate as the length of the period shrinks to zero, $\Delta t \to 0$.

In the limit, the growth rate is the derivative,

$$\lim_{\Delta t \to 0} \frac{1}{x} \frac{\Delta x}{\Delta t} = \frac{1}{x} \frac{\mathrm{d}x}{\mathrm{d}t}.$$

Derivative of the Logarithm

The growth rate is also the derivative of the logarithm,

$$\frac{\mathrm{d}\ln x}{\mathrm{d}t} = \frac{1}{x}\frac{\mathrm{d}x}{\mathrm{d}t}.$$

For a small changes, the change in the logarithm must be the fractional change,

$$\Delta \ln x \approx \frac{\Delta x}{x}.$$

Constant Growth Rate

If the variable grows at the constant rate g,

$$\ln x_t = \ln x_0 + gt.$$

Taking the exponent shows

$$x_t = \mathrm{e}^{\ln x_t} = \mathrm{e}^{(\ln x_0 + gt)} = \mathrm{e}^{\ln x_0}\mathrm{e}^{gt} = x_0\mathrm{e}^{gt}.$$

Thus

$$\mathrm{e}^{gt}$$

represents constant growth at rate g.

Graph

A non-logarithmic graph of $x_t = x_0 \mathrm{e}^{gt}$ against time has a steadily increasing slope.

A logarithmic graph of $\ln x_t = \ln \left(x_0 \mathrm{e}^{gt} \right) = \ln x_0 + gt$ against time has the constant slope g. Equal upward and downward shifts along the curve represent equal upward and downward percentage changes in the variable.

Growth Rate of a Product

The growth rate of a product is the sum of the growth rates.

For example, nominal income Y is the price level P times real income y,

$$Y = Py.$$

Then the growth rate of nominal income is the inflation rate plus the growth rate of real income,

$$
\begin{aligned}
\frac{1}{Y}\frac{\mathrm{d}Y}{\mathrm{d}t} &= \frac{\mathrm{d}\ln Y}{\mathrm{d}t} \\
&= \frac{\mathrm{d}\ln Py}{\mathrm{d}t} \\
&= \frac{\mathrm{d}\left(\ln P + \ln y\right)}{\mathrm{d}t} \\
&= \frac{\mathrm{d}\ln P}{\mathrm{d}t} + \frac{\mathrm{d}\ln y}{\mathrm{d}t} \\
&= \frac{1}{P}\frac{\mathrm{d}P}{\mathrm{d}t} + \frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}t}.
\end{aligned}
$$

Growth Rate of a Ratio

Analogously, the growth rate of a ratio is the growth rate of the numerator minus the growth rate of the denominator.

For example, the growth rate of the real wage $\frac{W}{P}$ is the growth rate of the nominal wage W minus the growth rate of the nominal price level P.