

### Question

Consider the Solow neoclassical one-sector growth model with rate of population growth  $n = .04$ , and aggregate production function

$$F(K, L) = \frac{KL}{K+L}.$$

(a) Find the intensive production function  $y = f(k)$ , output *per capita* as a function of the capital/labor ratio  $k = K/L$ . What is Solow's equation for the evolution of the capital/labor ratio?

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(b) If the saving fraction  $s = .20$  and the initial capital/labor ratio is 1, describe the long-run behavior of the economy. What is consumption *per capita* in the steady state?

(c) What is meant by the golden rule of saving? For the steady state, calculate the real interest rate  $r = f'(k)$ , and compare this value to  $n$ . Relate the steady state to the golden rule.

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### Answer

(a)

$$f(k) = \frac{F(K, L)}{L} = \frac{\left(\frac{KL}{K+L}\right)}{L} = \frac{K}{K+L} = \frac{L}{L} \frac{\left(\frac{K}{L}\right)}{\left(\frac{K}{L} + 1\right)} = \frac{k}{k+1}.$$

More directly,

$$f(k) = F(k, 1) = \frac{k}{k+1}.$$

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Solow's equation for the evolution of the capital/labor ratio is

$$\frac{dk}{dt} = sf(k) - nk.$$

The change in capital  $dk/dt$  (capital deepening *per capita*) is the difference between  $sf(k)$  (saving *per capita*) and  $nk$  (capital widening *per capita*).

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(b) In the long run, the economy converges to steady-state growth. To solve for the steady-state capital/labor, set  $dk/dt = 0$  and solve for  $k$ :

$$\begin{aligned} 0 &= sf(k) - nk \\ &= s \frac{k}{k+1} - nk. \end{aligned}$$

Cancelling  $k$  gives

$$0 = s \frac{1}{k+1} - n,$$

and solving gives

$$k = \frac{s}{n} - 1 = \frac{.20}{.04} - 1 = 4.$$

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If initially  $k = 1$ , it will gradually increase until  $k = 4$ . In the long run, output *per capita* is  $f(4) = \frac{4}{4+1} = .8$ . Saving *per capita* is  $sf(k) = .20 \times .8 = .16$ , and consumption *per capita* is  $.8 - .16 = .64$ . Population, capital, and output all grow at rate  $n = .04$ .

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(c) The golden rule of saving is to pick  $s$  to maximize steady state *per capita* consumption.

The following three conditions are equivalent:

- a) golden-rule saving rate;
- b) population growth = interest rate;
- c) saving rate = capital income share.

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The real interest rate is the marginal product of capital, the slope of the intensive production function:

$$\begin{aligned}
 r &= f'(k) \\
 &= \frac{1}{k+1} - \frac{k}{(k+1)^2} \\
 &= \frac{k+1}{(k+1)^2} - \frac{k}{(k+1)^2} \\
 &= \frac{1}{(k+1)^2} \\
 &= \frac{1}{(4+1)^2} \\
 &= .04.
 \end{aligned}$$

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Here we have the second condition,  $n = r$ , so  $s = .20$  is the golden rule of saving.

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Of course the third condition for the golden rule must also hold. Profit *per capita*

$$rk = .04 \times 4 = .16$$

equals saving *per capita*.

The capital income share is profit *per capita* divided by income *per capita*,

$$\frac{.16}{.80} = .20,$$

which is the saving rate.

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